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Lattice fuzzy transforms from the perspective of mathematical morphology

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Abstract

The compositions of direct and inverse fuzzy transforms constitute powerful tools in knowledge extraction and representation that have been applied to a large variety of problems in computational intelligence as well as in image processing and computer vision. Fuzzy transforms (FTs) have linear as well as lattice-based versions. In this paper, we extend the latter FTs, known as lattice FTs, and relate these operators and their underlying mathematical structures to the ones of mathematical morphology (MM), in particular to the ones of MM on complete lattices and $\mathbb{L}$-fuzzy MM.

Keywords: Lattice fuzzy transform; Residuated lattice; Mathematical morphology on complete lattices; $\mathbb{L}$-fuzzy mathematical morphology; Adjunction; Erosion; Dilation; Opening; Closing; Structuring element; Structuring function

1. Introduction

Fuzzy transforms [1,2] play a role that is similar to the ones of well-known transforms such as the Fourier, Laplace, Hilbert, Mellin, Z-, and wavelet transforms [3]. All of these transforms first map functions residing in an original space to functions in a transformed space before mapping them back into the original space. The first transformation is called the direct transform and the latter transformation is called the inverse transform.

In this paper, we generalize the definitions of lattice FTs by defining direct lattice FTs as (not necessarily discrete) mappings $\mathcal{F}_{\mathbb{L}}(\mathcal{P}) \to \mathcal{F}_{\mathbb{L}}(\mathcal{K})$ and inverse lattice FTs as mappings $\mathcal{F}_{\mathbb{L}}(\mathcal{K}) \to \mathcal{F}_{\mathbb{L}}(\mathcal{P})$, where $\mathcal{P}$ and $\mathcal{K}$ are arbitrary universes and $(\mathbb{L}, \vee, \wedge, \star, \to)$ is a residuated lattice [1,4,5]. In this context, recall that $\mathcal{F}_{\mathbb{L}}(X)$ denotes the class of $\mathbb{L}$-fuzzy sets [6]. We focus on the special case where the lattice $\mathbb{L}$ is complete [7].

The majority of transforms are linear in nature. Exceptions to this rule include the two pairs consisting of direct and inverse lattice FTs introduced by Perfilieva [1]. The existence of two pairs of lattice FTs is due to the duality principle for partially ordered sets which implies that for every operator in MM on complete lattices there is a dual operator [8]. More precisely, this paper demonstrates that one of the lattice FTs performs an erosion in the direct phase and a dilation in the inverse phase while the other one performs a dilation in the direct phase and an erosion in

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the inverse phase. Thus, the composition of a direct lattice FT and an inverse lattice FT either yields a closing or an opening in MM on complete lattices [9,10].

Direct and inverse lattice FTs do not only constitute erosions and dilations as defined via their algebraic properties in general MM on complete lattices but also erosions and dilations from a geometrical or topological point of view, since these operators can be expressed in $\mathbb{L}$-fuzzy MM [11] in terms of inclusion and intersection operators using $\mathbb{L}$-*fuzzy structuring functions* (SFs) that we define as elements of $(\mathcal{F}_{\mathbb{L}}(X))^Y$ for arbitrary universes X and Y. We also prove that every $\mathbb{L}$-fuzzy erosion and every $\mathbb{L}$-fuzzy dilation of an image by a (translation invariant) structuring element or an SF in $\mathbb{L}$-fuzzy MM can be written as a direct or inverse lattice FT.

The paper is organized as follows. Section 2 provides some lattice-theoretical background of MM. Section 3 reviews and extends some concepts and results of $\mathbb{L}$-fuzzy MM. Section 4 provides some links between the mathematical structures used in $\mathbb{L}$-fuzzy MM and the ones used in conjunction with lattice FTs. Section 5 relates direct and inverse lattice FTs to operators of MM, in particular to the ones of MM on complete lattices and $\mathbb{L}$-fuzzy MM. This section also includes some properties of lattice FTs as well as some remarks on the applicability of lattice FTs and their relationship to linear FTs. We finish with some concluding remarks.

2. Basic concepts of lattice theory for MM

Lattice theory has found applications in diverse areas such as mathematical morphology [11,12], fuzzy set theory [6], computational intelligence [13,14], automated decision making [15], and formal concept analysis [16] and the expression "lattice computing" emerged in recent years [13,17]. Note that lattice fuzzy transforms [1] can also be viewed as lattice computing techniques. Before exhibiting their relation to MM, we introduce some pertinent notations and provide a brief review and some extensions of lattice-theoretical concepts that play an important role in MM ever since complete lattices were established as an appropriate mathematical framework for (binary and gray-scale) MM [10,12,18,19].

Let $(\mathbb{L}, \leq)$ be a *lattice* [7]. If the partial order $\leq$ arises clearly from the context, then we simply write $\mathbb{L}$ instead of $(\mathbb{L}, \leq)$. The infimum and the supremum of $\{a, b\} \subseteq \mathbb{L}$ are respectively denoted by $a \wedge b$ and $a \vee b$. For any $M \subseteq \mathbb{L}$, we denote the infimum of M by the symbol $\bigwedge M$ and we write $\bigwedge_{j \in J} a^j$ instead of $\bigwedge M$ if $M = \{a^j : j \in J\}$ (here, j merely stands for an index and not for an exponent) for an index set J. Similarly, the symbols $\bigvee M$ and $\bigvee_{j \in J} a^j$ denote the suprema of M and $\{a^j : j \in J\}$, respectively.

Recall that a *bounded lattice* $\mathbb{L}$ contains both $\bigwedge \mathbb{L}$ and $\bigvee \mathbb{L}$ that are respectively denoted using the symbols $0_{\mathbb{L}}$ and $1_{\mathbb{L}}$. Given a bounded lattice $\mathbb{L}$, we refer to a decreasing and involutive mapping $\nu : \mathbb{L} \to \mathbb{L}$ as a *negation* on $\mathbb{L}$. This definition is the usual one in MM where a negation is involutive [18]. If $\bigwedge M, \bigvee M \in \mathbb{L}$ for all $M \subseteq \mathbb{L}$, then ones speaks of a *complete lattice* [7]. Consider $\mathbb{M} \subseteq \mathbb{L}$, where $(\mathbb{L}, \leq)$ is a complete lattice. If $(\mathbb{M}, \leq)$ represents a complete lattice as well then $\mathbb{M}$ is said to be a *complete sublattice* of $\mathbb{L}$.

If $\mathbb{L}$ is a lattice, a bounded lattice, or a complete lattice, then $\mathbb{L}^X$, i.e., the class of functions $X \to \mathbb{L}$, is also a lattice, a bounded lattice, or a complete lattice, respectively. Here, the partial order on $\mathbb{L}^X$ is given as follows:

$$f \leq g \Leftrightarrow f(x) \leq g(x) \; \forall x \in X. \tag{1}$$

Recall that an $\mathbb{L}$-fuzzy set A consists of a universe X together with a membership function $\mu_A : X \to \mathbb{L}$ [6]. If $\mathbb{L}$ is a complete lattice, then the class of $\mathbb{L}$-fuzzy sets over the universe X, denoted using the symbol $\mathcal{F}_{\mathbb{L}}(X)$, also represents a complete lattice with the following partial order:

$$A \leq B \Leftrightarrow \mu_A \leq \mu_B \; \forall A, B \in \mathcal{F}_{\mathbb{L}}(X). \tag{2}$$

In the special case where $\mathbb{L} = [0, 1]$, we obtain $\mathcal{F}_{\mathbb{L}}(X) = \mathcal{F}(X)$, i.e., the class of fuzzy sets over the universe X. If $A \in \mathcal{F}_{\mathbb{L}}(X)$, then we simply write $A(x)$ instead of $\mu_A(x)$ to denote the membership degree of $x \in X$ in A.

Given complete lattices $\mathbb{L}$ and $\mathbb{M}$, a bijection $\varphi : \mathbb{L} \to \mathbb{M}$ is called a *complete lattice isomorphism* if and only if φ represents an order-embedding [20], i.e., $\varphi(a) \leq \varphi(b)$ if and only if $a \leq b$. If there exists a complete lattice isomorphism $\varphi : \mathbb{L} \to \mathbb{M}$, then the (complete lattices $\mathbb{L}$ and $\mathbb{M}$ are said to be isomorphic and in this case we write $\mathbb{L} \simeq \mathbb{M}$. Given an arbitrary universe X, an isomorphism $\varphi : \mathbb{L} \to \mathbb{M}$ induces an isomorphism $\phi : \mathbb{L}^X \to \mathbb{M}^X$ that is given by:

$$(\phi(f))(\mathbf{x}) = \varphi(f(\mathbf{x})), \; \forall f \in \mathbb{L}^X, \mathbf{x} \in X. \tag{3}$$