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Generation of partial orders for intervals by means of the slope function

Jun Han [a], Bao Qing Hu [b,*]

[a] School of Science, Chongqing Jiaotong University, Chongqing, 400074, PR China
[b] School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, PR China

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Abstract

Fuzzy numbers used in mathematical programming in the form of interval coefficient, characterize the uncertainty resulting from imprecise, uncertain and incomplete data. And intervals are cut-sets of fuzzy numbers. Since interval ordering is undertaken prior to studying the comparability of intervals, the problem of ordering intervals is crucial for many applications. We introduce a new general parameterized relation between intervals and study the conditions under which it is a partial order, in order to generalize several classical partial orders with the help of the so-called slope function.

Keywords: Interval; Interval order relation; Boolean algebra

1. Introduction

Interval-valued fuzzy set [23], which is equivalent to intuitionistic fuzzy set [3], has been widely used in image processing, decision making, classification and so on. It is obvious that the interval ordering plays an important role in the arithmetic operation of intervals on the real line $\mathbb{R}$, especially on closed interval [0, 1]. Several order relations for intervals, the development of whose application has been witnessed by many publications such as [1,2,4,7–9,11–13, 15–21], have been introduced and discussed in [5,6,10,14,22].

On one hand, in 1990, Ishibuchi and Tanaka [14] introduced five partial orders $\leqslant_{LR}$, $\leqslant_{mw}$, $\leqslant_{cw}$, $\leqslant_{Lm}$ and $\leqslant_{Rm}$ for intervals, in order to convert the maximization problem with the interval objective function into a multi-objective problem using the order relations. In 1996, Chanas and Kuchta [6] generalized known concepts of the solution of the linear programming problem with interval coefficients in the objective function based on preference relations between intervals. And a whole family of preference relations, introduced by them, comprises some partial orders as a special case. On the other hand, in 2006, Hu and Wang [10] defined a total order $\leqslant_{HW}$, which can compare any two intervals, and various properties were established for solving problems with uncertainties. At the same time, Xu and Yager [22]

* Corresponding author.
E-mail addresses: guihai-4728@whu.edu.cn (J. Han), bqhu@whu.edu.cn (B.Q. Hu).

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gave a total order $\leqslant_{XY}$ which is different from $\leqslant_{HW}$ for some geometric aggregation operators based on intuitionistic fuzzy sets. A method to build the admissible orders in terms of two aggregation functions was proposed by Bustince et al. [5] in 2013. The admissible order is a total order and refines the partial order $\leqslant_{LR}$. And it is proved that some of the most used examples of total orders that appear in the literature are specific cases of their construction.

Let us recall these well-known concepts about order relations for intervals. We only consider the order relations of intervals on real number set $\mathbb{R}$. And the set of all intervals on $\mathbb{R}$ is denoted by $I_{\mathbb{R}}$. Let $A = [\underline{a}, \bar{a}]$ and $B = [\underline{b}, \bar{b}]$ be two intervals. The mid-point $m(A)$ and half-width $w(A)$ are defined respectively as follows.

$$m(A) \triangleq \frac{1}{2}(\underline{a} + \bar{a}),$$
$$w(A) \triangleq \frac{1}{2}(\bar{a} - \underline{a}).$$

And these different methods of ranking intervals are shown as follows.

(1) $A \leqslant_{LR} B$ iff $\underline{a} \leqslant \underline{b} \wedge \bar{a} \leqslant \bar{b}$ (see [14]);
(2) $A \leqslant_{mw} B$ iff $m(A) \leqslant m(B) \wedge w(A) \geqslant w(B)$ (see [14]);
(3) $A \leqslant_{cw} B$ iff $m(A) \leqslant m(B) \wedge w(A) \leqslant w(B)$ (see [14]);
(4) $A \leqslant_{Lm} B$ iff $m(A) \leqslant m(B) \wedge \underline{a} \leqslant \underline{b}$ (see [14]);
(5) $A \leqslant_{Rm} B$ iff $m(A) \leqslant m(B) \wedge \bar{a} \leqslant \bar{b}$ (see [14]);
(6) $A \leqslant_{HW} B$ iff $m(A) < m(B) \vee (m(A) = m(B) \wedge w(A) \geqslant w(B))$ (see [10]);
(7) $A \leqslant_{XY} B$ iff $m(A) < m(B) \vee (m(A) = m(B) \wedge w(A) \leqslant w(B))$ (see [22]);
(8) $A \leqslant_{Lex1} B$ iff $\underline{a} < \underline{b} \vee (\underline{a} = \underline{b} \wedge \bar{a} \leqslant \bar{b})$ (see [5]);
(9) $A \leqslant_{Lex2} B$ iff $\bar{a} < \bar{b} \vee (\bar{a} = \bar{b} \wedge \underline{a} \leqslant \underline{b})$ (see [5]);
(10) $A \leqslant_{\alpha+} B$ iff $\alpha < \beta \wedge (K_\alpha(A) < K_\alpha(B) \vee (K_\alpha(A) = K_\alpha(B) \wedge K_\beta(A) \leqslant K_\beta(B)))$ (see [5]);
(11) $A \leqslant_{\alpha-} B$ iff $\alpha > \beta \wedge (K_\alpha(A) < K_\alpha(B) \vee (K_\alpha(A) = K_\alpha(B) \wedge K_\beta(A) \leqslant K_\beta(B)))$ (see [5]);

where K_α and K_β are defined as

$$K_\alpha(A) = \underline{a} + \alpha(\bar{a} - \underline{a}) = (1 - \alpha)\underline{a} + \alpha\bar{a}, \quad \alpha \in [0, 1],$$
$$K_\beta(A) = \underline{a} + \beta(\bar{a} - \underline{a}) = (1 - \beta)\underline{a} + \beta\bar{a}, \quad \beta \in [0, 1].$$

In our opinion the above orders work in a similar way although they are defined from various angles, so we need a tool to unify them by putting them in the same frame. Considering those classical orders and comparing the left endpoints of any two intervals, we consider the above classical orders in three aspects and then a new binary relation is constructed in the form of $\leqslant_{U,V,W}$ by the slope function $k(A, B)$. What is more, the binary relation $\leqslant_{U,V,W}$ can be proved to be a generalization of several classical interval order relations.

2. A new binary relation and its algebraic properties

In this section, we try to introduce a new binary relation. And it is used to encompass the above orders as some specific cases in the next section. Since the application of these interval orderings is discussed in [1,2,4,7–9,11–13,15–21], our approach covers those methods.

In order to find certain qualities in common among those orders on $I_{\mathbb{R}}$, we consider the first three partial orders. For any given $a, b, c \in \mathbb{R}$, if $a < b < c$, then we can get six different intervals

$$[a, a], [a, b], [a, c], [b, b], [b, c], [c, c].$$

And we write $P \triangleq \{[a, a], [a, b], [a, c], [b, b], [b, c], [c, c]\}$. To explain more clearly, the Hasse diagrams of the partial order set $\langle P, \leqslant_{LR}\rangle$, $\langle P, \leqslant_{mw}\rangle$ and $\langle P, \leqslant_{cw}\rangle$ are shown in Fig. 1.

It is clear that these diagrams can be seen as three different positions of one triangle in various angles. What is more, it is easy to see that all the arrows point to the similar direction (see Fig. 2), if we draw an arrow from one node to another while the latter covers the former in Hasse diagram.