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On the stability of fuzzy dynamical systems

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Abstract

In this work we study the asymptotic behavior of fuzzy solutions obtained by using Zadeh's extension at the deterministic solutions of initial value problems. We obtain some result regarding the existence of fuzzy equilibrium points that generalize the already known results. As show earlier, the membership function of the fuzzy equilibrium points may be obtained in a relatively simple way. Also, we study the projection of the fuzzy solution on the appropriate subspaces of the phase space. Several examples are given to illustrate the obtained results.

Keywords: Fuzzy sets; Equilibrium points; Dynamical systems

1. Introduction

Fuzzy differential equations result from the need to consider certain types of subjectivity in the state variables of dynamic systems, modeled by deterministic differential equations. Fuzzy initial value problems, denoted by

$$\frac{d\boldsymbol{x}}{dt} = F\big(\boldsymbol{x}(t)\big), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_o \in \mathcal{F}(U), \quad U \subseteq \mathbb{R}^n \tag{1}$$

have several distinct interpretations. Some authors use the H-derivative from the variation process of $\boldsymbol{x}(t)$ [1–13]. Others construct the solution of (1) by using a family of differential inclusions [14–17]. Some other interpretations are presented in [18–20].

In the recent past, Zadeh's extension principle has been used to obtain solutions of (1) [21–24,32–37], in the following way: If the fuzzy function $F : \mathcal{F}(U) \to \mathcal{F}(U)$, is the obtained by Zadeh extension of a continuous function $f : U \to U$ then a solution of (1) is defined as Zadeh's extension, on initial condition, of the deterministic solution $\varphi_t(x_o)$, of the associated initial value problem

$$\frac{dx}{dt} = f\big(x(t)\big), \qquad x(0) = x_o \in U. \tag{2}$$

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This fuzzy solution of Eq. (1) is denoted by $\hat{\varphi}_t(\boldsymbol{x}_o)$. Under appropriate conditions the different interpretations of Eq. (1) coincide [23,25,26].

The authors of [24] have studied the existence of equilibrium points and the stability of steady state of the fuzzy solution $\hat{\varphi}_t(\boldsymbol{x}_o)$. According to them, equilibrium points for deterministic equations characterize crisp equilibrium points for the fuzzy equation and the type of stability of such points coincide in the corresponding metrics.

In this work we present some new properties of the fuzzy solutions $\hat{\varphi}_t(\boldsymbol{x}_o)$. We also seek to generalize some result of fuzzy equilibrium points presented in [24].

When the convergence of a deterministic solution to the equilibrium point depends on the initial condition or parameters, it is possible to show that the fuzzy solution converge to a fuzzy equilibrium point, and we show how to determine the membership function of such an equilibrium point. Also, we study the projection of the fuzzy solution on the space of the phase space. By that projections we may identify the behavior of each component of the fuzzy solution as a function of time. For unidimensional equations we present an even simpler way to identify the α-cuts of such projections. Various examples are given in order to clarify the new results obtained.

This paper is organized as follows. In Section 2 we present some basic results of both deterministic differential equations and fuzzy sets. In Section 3 we seek the conditions for the existence and stability of equilibrium points for the fuzzy equation. In Section 4 we consider the projections of the fuzzy solution on the subspaces of $\mathcal{F}(U)$.

2. Basic concepts

2.1. Deterministic solutions

Let $\varphi_t : U \to U$ be a flow of Eq. (2). In this work, for the sake of analysis, we assume that the deterministic solution is defined for all $t \in \mathbb{R}_+$ (see [27]).

When the function $f : U \to \mathbb{R}^n$ in (2) is C^1 and has constant of Lipschitz $K > 0$, the deterministic solution satisfies the inequality

$$\left\|\varphi_t(x_o) - \varphi_t(y_o)\right\| \leqslant \|x_o - y_o\| \, \mathrm{e}^{Kt}, \tag{3}$$

for all $t \in \mathbb{R}_+$ (see [28]).

The equilibrium solution is very important to the qualitative analysis of deterministic solutions of nonlinear equations. Let x_e an equilibrium point for Eq. (2), so $f(x_e) = 0$. If the Jacobian matrix of f, calculated at x_e, has eigenvalues with negative real parts, then there is a neighborhood $V \subset U$ of x_e such that, for all $x_o \in V$, the deterministic solution $\varphi_t(x_o)$ exists for all $t \in \mathbb{R}_+$ satisfying

$$\left\|\varphi_t(x_o) - x_e\right\| \leqslant B\|x_o - x_e\| \, \mathrm{e}^{-bt}, \tag{4}$$

for the positive constants B and b [28]. An equilibrium point with this property is frequently called *exponential attractor*.

The *region of attraction* of an equilibrium point $x_e \in U$ is the set, $A(x_e) \subset U$, defined by

$$A(x_e) = \left\{x_o \in \mathbb{R}^n \colon \varphi_t(x_o) \to x_e, \ t \to \infty\right\}.$$

For asymptotically stable equilibrium points, we have the following theorem.

Theorem 1. *(See [29].) Let x_e be an asymptotically stable equilibrium points of Eq. (2). If $K \subset A(x_e)$ is compact, then*

$$\operatorname{dist}\bigl(\varphi_t(K), x_e\bigr) = \sup_{x_o \in K} \left\|\varphi_t(x_o) - x_e\right\| \to 0$$

when $t \to \infty$.

In other words, the last proposition guarantees that an asymptotically stable equilibrium point attracts compact subsets of region of attraction.

We now consider solutions of autonomous differential equations that depends on parameters. That is, solutions of equations of the form