Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 242 (2014) 31–55

FUZZY sets and systems

www.elsevier.com/locate/fss

Type ⟨1, 1⟩ fuzzy quantifiers determined by fuzzy measures on residuated lattices. Part I. Basic definitions and examples ☆

Antonín Dvořák *, Michal Holčapek

National Supercomputing Center IT4Innovations, Division of University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic

Available online 17 October 2013

Abstract

We study fuzzy quantifiers of type ⟨1, 1⟩ defined using fuzzy measures and integrals. Residuated lattices are considered as structures of truth values. We present basic notions on fuzzy measures over algebras of fuzzy subsets of a given fuzzy set, and the definition and necessary properties of so-called $\odot$-fuzzy integrals. Residuated lattice operations are established to derive operations between fuzzy sets describing degrees in which formulas expressing relations between fuzzy sets are true. Finally, a general definition of type ⟨1, 1⟩ fuzzy quantifiers determined by fuzzy measures and integrals is introduced and several examples of important natural language quantifiers are modeled using this approach.

Keywords: Fuzzy measure; Fuzzy integral; Fuzzy logic; Fuzzy quantifier

1. Introduction

We propose a general model of fuzzy quantifiers (with two arguments) determined by fuzzy measures and integrals. Besides its potential for use in reasoning (a logical theory for these quantifiers in line with previous studies [27,28] is in preparation), it can also be used in various applications. Using various fuzzy measure spaces, we cover a wide class of fuzzy quantifiers discussed in the literature (see the examples in Section 5). However, in this series of papers we are mainly interested in establishing a theoretical framework for a sufficiently general class of fuzzy quantifiers that can be tailored to individual applications with important theoretical properties guaranteed.

The study of generalized quantifiers evolved from pioneering work by Mostowski [26], Lindström [23], Barwise and Cooper [1], and van Benthem [39] into a large research field. Peters and Westerståhl provide an overview of the field and have described many results [34]. In the classical setting, a quantifier Q of type ⟨1, 1⟩ (such as "every", "many", etc.) is usually modeled, given a universe M, as a mapping Q_M from the Cartesian product of power sets $\mathcal{P}(M) \times \mathcal{P}(M)$ to the set {*false*, *true*} or, equivalently, as a binary relation on subsets of M.

As previously discussed [11], when we think about the definition and properties of generalized quantifiers such as *many*, *few*, and others, we feel that their truth values should not change abruptly if we gradually change the cardinality

☆ This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070).

* Corresponding author. Tel.: +420 597091406; fax: +420 596120478.
E-mail addresses: antonin.dvorak@osu.cz (A. Dvořák), michal.holcapek@osu.cz (M. Holčapek).

0165-0114/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.fss.2013.10.003

of the corresponding sets of objects. Therefore, researchers started to consider more than two truth values in this context and so-called *fuzzy quantifiers* emerged. Starting from a generalization of the definition from the previous paragraph, instead of the set {*false*, *true*}, some other truth value structure is considered, notably the interval [0, 1] of real numbers, and more general algebraic structures (residuated lattices, MV-algebras).[1]

Research in the field of fuzzy quantifiers started with work by Zadeh [41,42] and was continued by Ralescu [37], Thiele [38] and others [24,25,29]. Dubois et al. reported important results on syllogisms involving vaguely specified numerical quantifiers [8] and on automated reasoning in probabilistic knowledge bases with probabilities expressed as quantifiers [7]. The authors proposed use of the quantifier "most" for computation of the quotient operator in fuzzy relational databases [9]. Bosc et al. studied an application of Zadeh's fuzzy quantifiers and Yager's OWA operators [40] in evaluating fuzzy quantified queries in databases [3]. An interesting interpretation of the quantifier "most" in the same context involves neglecting one or several of the worst-satisfied cases [5]. Bosc et al. proposed use of the Sugeno fuzzy integral for interpretation of flexible queries in SQL-like languages [4]. A generalization of an OWA operator has been suggested for interpretation of non-monotonic quantifiers such as "about half" [22].

An important contribution by Hájek is consideration of the quantifier "many" using relative frequencies [17]. He pointed out interconnections between generalized quantifiers and modalities. A comprehensive study of fuzzy quantifiers was undertaken by Glöckner [14,15]. Novák studied so-called intermediate quantifiers [32], mainly from a syntactic point of view in the frame of fuzzy type theory [30]. Generalized Aristotelian syllogisms [35] were syntactically proved by Murinová and Novák using a slightly different definition of fuzzy quantifiers than ours [27].

Here we follow a research line originated by Glöckner [14] and elaborated in [18]. We previously studied fuzzy quantifiers of type $\langle 1 \rangle$ (with one argument) [11],[2] which denote important noun phrases of natural language such as something in "Something is broken", everyone in "Everyone likes Bob", and nobody in "Nobody knows everything". Classical logical quantifiers such as "for all" and "there exists" also belong to this type. A natural extension of this research is to study quantifiers of type $\langle 1, 1 \rangle$ (e.g. every in "Every book has leaves", most in "Most birds fly") that take two arguments. It has been suggested that these type $\langle 1, 1 \rangle$ quantifiers are the most important from the point of view of natural language semantics [34]. The reason is that two-argument quantifiers are most common in natural language usage. Moreover, they can often be used to express or decompose quantifiers of other types.

In this paper we extend our modeling of fuzzy quantifiers determined by fuzzy measures to the $\langle 1, 1 \rangle$ type; preliminary results have been reported elsewhere [12]. In this case it would be advantageous to work with a slightly different definition of fuzzy measures and integrals. The first argument of a $\langle 1, 1 \rangle$ quantifier is called *restriction* and the second is *scope*; for example, in "Every book has leaves", to be a book is the restriction and to have leaves is the scope. It is natural to think of the restriction as a *new universe* for the quantifier (in our example, to determine the truth value, only objects fulfilling the restriction condition are important, i.e., books). Because we are working with fuzzy subsets of some universe M, we should be able to define quantifiers on *fuzzy universes*. Therefore, we introduced a new type of fuzzy measure space defined on algebras of subsets of a *fuzzy* set A and a corresponding fuzzy integral, the so-called $\odot$-fuzzy integral (Section 2.3) [13].

Why do we think that our $\odot$-fuzzy integral is an appropriate tool for modeling of natural language quantifiers? We previously argued that a possible logical analysis of a sentence such as "Many sportsmen are tall" is as follows [13]: we search for a fuzzy subset of the fuzzy set of sportsmen that is large (i.e., its measure is as great as possible) and for all elements x from its support it holds that if x is a sportsman, then x is tall. This leads to the second-order formula

$$\mathsf{many}(Sp, Ta) := \big(\exists Y \in \mathcal{F}^{-}_{Sp}\big)\big(\forall x \in \mathrm{Supp}(Y)\big(\mu(Y) \mathbin{\&} \big(Sp(x) \Rightarrow Ta(x)\big)\big)\big), \tag{1}$$

where Sp and Ta denote fuzzy sets of sportsmen and tall people, respectively, $\mathcal{F}^{-}_{Sp}$ is the set of all non-empty fuzzy subsets of fuzzy set Sp, $\mathrm{Supp}(Y)$ is the support of fuzzy set Y, and μ denotes a measure of fuzzy sets. The semantic counterpart of this formula is exactly the $\odot$-integral of the fuzzy set $Sp \to Ta$, where $\to$ is the operation of residuum that models the implication.

There is often a distinction made between crisp quantification of fuzzy predicates, which assigns a value from {*false*, *true*} to a pair of fuzzy sets, and fuzzy quantification, in which quantifiers are represented by fuzzy sets [25].

[1] Sometimes [25], the term "fuzzy quantifier" is reserved for quantifiers represented by fuzzy sets. In this paper, we understand (for type $\langle 1, 1 \rangle$) a fuzzy quantifier to be any mapping assigning a truth value to a pair of fuzzy sets (Definition 4.1).

[2] The notation type $\langle 1 \rangle$ and type $\langle 1, 1 \rangle$ originated in [23], where quantifiers are understood to be classes of relational structures of a certain type (representing a number of arguments and variable binding). It is widely used in the literature on generalized quantifiers [19,34].