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Interpolativity of at-least and at-most models of monotone fuzzy rule bases with multiple antecedent variables ☆

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Abstract

Of the many desirable properties of fuzzy inference systems, not all are known to co-exist. For instance, a fuzzy inference system should be interpolative, i.e., if a fuzzy set equivalent to one of the antecedent fuzzy sets appears on the input of such a system, the inferred output should be equivalent to the respective consequent fuzzy set. Similarly, the defuzzified output obtained from a system based on a monotone fuzzy rule base should be monotone, i.e., if x, x' are crisp inputs to the system that are ordered $x \leq x'$, then, the corresponding defuzzified outputs from the system y, y' should also be ordered accordingly, i.e., $y \leq y'$. However, the particular setting that ensures monotonicity need not simultaneously ensure interpolativity. Recently, Štěpnička and De Baets have investigated and demonstrated the co-existence of the above two properties in the case of fuzzy relational inference systems and single-input-single-output (SISO) rule bases. An extension of these results to the multiple-input-single-output (MISO) case is not straightforward due to the lack of a natural ordering in higher dimensions. In this work, we study the MISO case and show that similar results can be obtained when the monotone rule base is modeled based on at-most and at-least modifiers.

Keywords: Interpolativity; Fuzzy relational equations; At-least and at-most modifiers; Fuzzy rule-based systems; Monotonicity

1. Introduction

There are many desirable properties of fuzzy inference systems and it is crucial to ask, whether they co-exist or not. The interpolativity, i.e., the preservation of the modus ponens property, is an essential property being very well explored and accepted by the research community. It can be roughly described in the following way: if a fuzzy set equivalent to one of the antecedent fuzzy sets appears on the input of such a system, the inferred output should be

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equivalent to the respective consequent fuzzy set. Another, less commonly studied yet not less important, property is the preservation of the monotonocity by an inference system equipped with a defuzzification. This property can be roughly described in the following way: if we consider a "monotone fuzzy rule base", the resulting function obtained by the application of the defuzzification should be monotone too. However, preservation of the monotonicity requires to use specific models of a given fuzzy rule base. Is the use of such specific models of monotone fuzzy rule bases consistent with the inteporlativity property and thus, will the preservation of the modus ponens and the preservation of the monotonocity co-exist under some reasonable and not very restrictive conditions?

This is the question that is addressed in this article for the multiple antecedent case, following the positive answer provided for the single input case in [1].

1.1. Fuzzy relational models of fuzzy inference systems

A fuzzy inference mechanism can be viewed as a mapping which – with the help of a model of a given fuzzy rule base – gives a meaningful output from basically imprecise inputs modeled by fuzzy sets. Of the many types of inference mechanisms and fuzzy rule base models proposed in fuzzy logic, we highlight the fuzzy relational approach, which is very common in the literature. The basic principles of such an approach consist in using an appropriate fuzzy relation to model a given fuzzy rule base and modeling the fuzzy inference mechanism by using an appropriate image of a fuzzy set under a fuzzy relation.

In this article, we restrict our focus to the fuzzy relational approach and we omit from our consideration many other approaches e.g. based on similarity based reasoning principles [2–4] or interpolative reasoning for sparse rule bases [5,6] although we are aware that these approaches under certain assumptions may become equivalent to the fuzzy relational ones [7].

Consider a fuzzy rule base containing rules of the following form:

$$\textbf{IF } X \textbf{ is } A_i \textbf{ THEN } Y \textbf{ is } B_i\ , \tag{1}$$

with X being a variable over a universe U, Y being a variable over a universe V, and A_i and B_i $(i = 1, \dots, n)$ being fuzzy sets over U and V, respectively. Such a fuzzy base is modeled by a fuzzy relation on $U \times V$.

Let us fix the used operations, particularly, let $*$ be a left-continuous t-norm and $\to$ its adjoint residuum (also residual fuzzy implication [8]). Note that this setting forms a residuated lattice $\mathcal{L} = \langle [0, 1], \wedge, \vee, *, \to, 0, 1\rangle$ whose properties we may freely use in the paper. Furthermore, we denote the set of all fuzzy sets on U by $\mathcal{F}(U)$, i.e.,

$$\mathcal{F}(U) = \{C \mid C : U \to [0, 1]\}.$$

There are two standard approaches to model fuzzy rule base (1) by a fuzzy relation on $U \times V$. The first one uses $\hat{R} \in \mathcal{F}(U \times V)$ defined by

$$\hat{R}(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))\ , \tag{2}$$

which reflects the conditional nature of the rules; another popular alternative, often called the Mamdani–Assilian model, [9,10] is the fuzzy relation $\check{R} \in \mathcal{F}(U \times V)$ defined by

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y))\ . \tag{3}$$

As recalled above, within the fuzzy relational framework, the fuzzy inference mechanism is mathematically nothing else but a way of computing an *image* of a fuzzy set under a fuzzy relation. The appropriate images are derived from *fuzzy relational compositions* [11,12]. Zadeh's Compositional Rule of Inference (CRI) [13], denoted by $\circ$ (based on the direct image [1]), is the most often used and well-established relational fuzzy inference mechanism. However, Pedrycz [14] proposed to use the Bandler–Kohout Subproduct (BKS) [15,16], denoted by $\lhd$ (based on the subdirect image), and Štěpnička with Jayaram [17] showed that this choice is fully comparable with the CRI, in terms of the preservation of important or advantageous properties.

We recall that, given a fuzzy set $A \in \mathcal{F}(U)$ and a fuzzy relation $R \in \mathcal{F}(U \times V)$, these images are the fuzzy sets on V defined by