Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 247 (2014) 1–17

FUZZY sets and systems

www.elsevier.com/locate/fss

# Validity in a logic that combines supervaluation and fuzzy logic based theories of vagueness

Sebastian Krinninger [1]

*University of Vienna, Faculty of Computer Science, Währinger Straße 29, 1090 Wien, Austria*

Received 5 April 2012; received in revised form 2 December 2013; accepted 5 December 2013

Available online 11 December 2013

**Abstract**

Supervaluationism and fuzzy logic are two complementary formalisms for reasoning with vague information. We study a framework for combining both approaches. Supervaluationism is modeled by a space of precisifications, essentially a Kripke structure. We equip this space with a probability measure to extract the truth value of each propositional variable by measuring the set of precisifications in which it is true. Complex formulas are evaluated by the truth functions given by a continuous t-norm and its residuum. We also add a universal modality to this logic. Besides unrestricted probability measures, we motivate two other natural classes: strictly positive and uniform probability measures. The goal of this paper is to analyze how the choice of a probability measure and a t-norm affects the set of valid formulas in our hybrid logic.


*Keywords:* Vagueness; Supervaluationism; t-norm based logics; Mathematical fuzzy logic; Non-classical logics

## 1. Introduction

Reasoning with vague information is one of the main motivations for fuzzy logic. Another approach for this purpose is supervaluationism and originates in the vagueness discourse in analytic philosophy. Fuzzy logic and supervaluationism follow very different principles. It seems natural to combine these complementary concepts of vagueness to a common framework. In this paper, we study certain aspects of such a framework.

Fuzzy logics are a class of truth-functional logics with the unit interval [0, 1] as the set of truth values. Following Hájek's approach of mathematical fuzzy logic [1], we consider logics that have a continuous t-norm as the truth function for conjunction and the corresponding residuum as the truth function for implication. Thus, the choice of a t-norm fully specifies a logic.

The baseline of supervaluationism is that a vague statement should be considered true if it is true for all ways of making it completely precise. Therefore, a vague situation is modeled by a space of precisifications. In every precisification of the space, statements are classically true or false. A person that for example is a borderline case of tallness would be considered tall in some precisifications and not tall in others. The truth in the precisifications

*E-mail address:* sebastian.krinninger@univie.ac.at.

[1] This research was partially supported by ESF/FWF Grant I143-G15 (LogICCC/LoMoReVI). It was partially conducted while the author was at the Vienna University of Technology, Austria.

0165-0114/$ – see front matter 
http://dx.doi.org/10.1016/j.fss.2013.12.002

should be in accordance with the intuitive use of language: if in a precisification a person with a height of 180 cm is considered tall, then also a person with a height of 190 cm should be considered tall. The supervaluationist's notion of truth is supertruth, which is defined as truth in all precisifications. Note that this model ultimately leads to a Kripke semantics and thus supervaluational logics are usually modal logics.

In this paper, we consider a certain approach of combining supervaluation and fuzzy logics. We extract the truth values of atomic formulas from the Kripke structure of the supervaluational model by equipping it with a probability measure. Complex formulas are interpreted according to the truth functions given by a continuous t-norm. Furthermore, a supertruth operator is added to express truth in all precisifications. Even in this simple framework some natural variations of our combination scheme arise. We could demand that no precisification is measured with 0 or that every precisification is measured uniformly. Thus, there is a certain design choice on how exactly both approaches should be combined. This can be compared to the situation for fuzzy logics where the choice of a t-norm determines properties of the resulting logic. In this paper, we analyze how the choice of the probability measure and the t-norm affects the validity of formulas.

Since the purpose of this paper is to study the effects of *combining* supervaluation and fuzzy logic, we will only work in the simplest possible setting. We restrict ourselves to the propositional level and only consider continuous t-norms and their residua in the truth value interval $[0, 1]$. This means that we do not consider left-continuous t-norms [2] or other generalizations nor any algebraic semantics. Concerning the supervaluational side, we do not impose any accessibility relations on the Kripke structures and assume that the space of precisifications is countable.

### *1.1. Further motivation*

The supervaluational approach towards vagueness is largely motivated by modeling *penumbral connections*. Fine [3] explains that a penumbral connection is a logical relation that holds among indefinite sentences. Fine's example is a (monochrome) blob whose color is at the borderline of red and pink. He argues that the sentence "the blob is red and pink" should be completely false because there can only be one color assigned to the blob. In a truth-functional approach, as for example fuzzy logic, one would usually assign an intermediate truth value, say 0.5, to the sentences "the blob is red" and "the blob is pink". The conjunction of these two sentences would then also receive an intermediate truth-value larger than 0. In the precisification-space approach, all vagueness is resolved in the precisifications. Therefore, in each precisification, exactly one of both sentences is true and the other one is false. Thus, the conjunction "the blob is red and pink" is false in each precisification, i.e., superfalse, which captures Fine's intuition regarding this penumbral connection. Observe also that all classical tautologies are preserved under supertruth.

Supervaluation only needs the qualitative information whether a sentence is true in all precisifications. In the hybrid approach, that was introduced by Fermüller and Kosik [4] and is also pursued in this paper, we additionally want to use the quantitative information conveyed by a precisification space. Intuitively, it should make a difference whether a sentence like "the blob is red" (which is not supertrue) is true in *some* or *almost all* precisifications. In a finite precisification space, this motivates the definition of the truth degree of a sentence as the relative frequency of those precisifications in which the sentence is true. If the sentence "the blob is red" is true in one half of the precisifications and false in the other half, then its truth degree should be 0.5.

As a natural generalization of this idea of extracting truth degrees we can assign weights to the precisifications. This leads to an additive measure whose value for the total space is 1, i.e., a probability measure. Following [5], using a probability measure for this purpose can be motivated as follows. Consider a function $\mu$ that assigns to every sentence $\varphi$ a value $\mu(\varphi)$ which is the degree of belief of a rational agent that $\varphi$ is true. If $\mu(\varphi)$ really represents this degree of belief, the agent should be willing to accept any bet of the form $(\alpha, \mu(\varphi), \varphi)$ where he has to pay $\alpha\mu(\varphi)$ and receives $\alpha$ if the sentence $\varphi$ is true and 0 otherwise. In fact, we now describe a situation where the agent will certainly not accept a sequence $(\alpha_i, \mu(\varphi_i), \varphi_i)_{1\leqslant i\leqslant m}$ of $m$ such bets. Accepting the bet $(\alpha_i, \mu(\varphi_i), \varphi_i)$ means that the agent has to pay $\alpha\mu(\varphi_i)$ and, in a precisification $s$, gains $\alpha$ if $\varphi$ is true in $s$ and 0 otherwise. Thus, the payoff in precisification $s$ is $\alpha_i(\|\varphi_i\|_s - \mu(\varphi_i))$ for the $i$-th bet, where $\|\varphi\|_s$ is the (classical) truth value of $\varphi$ in precisification $s$, and $\sum_i \alpha_i(\|\varphi_i\|_s - \mu(\varphi_i))$ in total. If $\sum_i \alpha_i(\|\varphi_i\|_s - \mu(\varphi_i)) < 0$ for *every* precisification $s$, then the sequence of bets $(\alpha_i, \mu(\varphi_i), \varphi_i)_{1\leqslant i\leqslant m}$ is called a *Dutch book*. A Dutch book implies sure loss for all ways of resolving the vagueness in a precisification space and therefore a rational agent will not accept it. Thus, we are only interested in functions $\mu$ that return degrees of belief such that no Dutch book exists. By de Finetti's well-known result [6] we know that there