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Representations through a monoid on the set of fuzzy implications

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Abstract

Fuzzy implications are one of the most important fuzzy logic connectives. In this work, we conduct a systematic algebraic study on the set $\mathbb{I}$ of all fuzzy implications. To this end, we propose a binary operation, denoted by $\circledast$, which makes $(\mathbb{I}, \circledast)$ a non-idempotent monoid. While this operation does not give a group structure, we determine the largest subgroup $\mathbb{S}$ of this monoid and using its representation define a group action of $\mathbb{S}$ that partitions $\mathbb{I}$ into equivalence classes. Based on these equivalence classes, we obtain a hitherto unknown representations of the two main families of fuzzy implications, viz., the f- and g-implications.

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1. Introduction

Fuzzy implications, along with triangular norms (t-norms, in short) form the two most important fuzzy logic connectives. They are a generalization of the classical implication and conjunction, respectively, to multi-valued logic and play an equally important role in fuzzy logic as their counterparts in classical logic.

Definition 1.1. (See [3], Definition 1.1.1.) A binary operation I on $[0, 1]$ is called a *fuzzy implication* if

(i) I is decreasing in the first variable and increasing in the second variable,
(ii) $I(0, 0) = I(1, 1) = 1$ and $I(1, 0) = 0$.

The set of all fuzzy implications will be denoted by $\mathbb{I}$. Table 1 (see also [3]) lists some examples of basic fuzzy implications.

Fuzzy implications have many applications in various fields like fuzzy control, approximate reasoning, decision making, fuzzy logic, etc. Their applicational value has been the *raison d'être* for more than three decades of research

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Table 1
Examples of fuzzy implications.

Name	Formula
Łukasiewicz	$I_{\mathbf{LK}}(x,y)=\min(1,1-x+y)$
Gödel	$I_{\mathbf{GD}}(x,y)=\begin{cases}1, & \text{if } x\leqslant y\\ y, & \text{if } x>y\end{cases}$
Reichenbach	$I_{\mathbf{RC}}(x,y)=1-x+xy$
Kleene–Dienes	$I_{\mathbf{KD}}(x,y)=\max(1-x,y)$
Goguen	$I_{\mathbf{GG}}(x,y)=\begin{cases}1, & \text{if } x\leqslant y\\ \frac{y}{x}, & \text{if } x>y\end{cases}$
Rescher	$I_{\mathbf{RS}}(x,y)=\begin{cases}1, & \text{if } x\leqslant y\\ 0, & \text{if } x>y\end{cases}$
Yager	$I_{\mathbf{YG}}(x,y)=\begin{cases}1, & \text{if } x=0 \text{ and } y=0\\ y^x, & \text{if } x>0 \text{ or } y>0\end{cases}$
Weber	$I_{\mathbf{WB}}(x,y)=\begin{cases}1, & \text{if } x<1\\ y, & \text{if } x=1\end{cases}$
Fodor	$I_{\mathbf{FD}}(x,y)=\begin{cases}1, & \text{if } x\leqslant y\\ \max(1-x,y), & \text{if } x>y\end{cases}$
Smallest FI	$I_{\mathbf{0}}(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or } y=1\\ 0, & \text{if } x>0 \text{ and } y<1\end{cases}$
Largest FI	$I_{\mathbf{1}}(x,y)=\begin{cases}1, & \text{if } x<1 \text{ or } y>0\\ 0, & \text{if } x=1 \text{ and } y=0\end{cases}$
Most strict FI ([21])	$I_{\mathbf{D}}(x,y)=\begin{cases}1, & \text{if } x=0\\ y, & \text{if } x>0\end{cases}$

on these operations and have made it essential to study various aspects of fuzzy implications in depth. Their analytical properties like continuity, intersections of families of fuzzy implications, relationship between the properties, etc., have been studied extensively in a comprehensive manner (see, for instance, the research monograph of Baczyński and Jayaram [3]).

An algebraic study of fuzzy implications can be done along the following lines:

(i) Let $\mathcal{L}$ denote the underlying set from which fuzzy propositions can assume their truth values. Usually, $\mathcal{L}=[0,1]$ or at least a poset. Then one imposes some axioms or properties on the fuzzy implication $\longrightarrow$ and studies the logics obtained or the equivalent algebras generated. For instance, see [5,19,10,6,7].

(ii) One can also define a *closed* binary operation $\circledast$ on the set $\mathbb{I}$ and study the algebraic structures obtained on it. For instance, let the operation $\circledast$ be the lattice operation of pointwise meet $\wedge$ or join $\vee$. From the view point of abstract algebra, we obtain that $(\mathbb{I},\wedge)$ and $(\mathbb{I},\vee)$ are commutative, integral, idempotent monoids.

1.1. Motivation for this work

In this work, we take the second of the above two approaches. Note that such a study would have two important ramifications.

(A) Firstly, since $(\mathbb{I},\circledast)$ is closed, given two fuzzy implications $I_1, I_2\in\mathbb{I}$, $I_1\circledast I_2\in\mathbb{I}$ and hence gives a way of generating new fuzzy implications from given ones.

(B) Secondly, if one were able to impose a richer algebraic structure, say $(\mathbb{I},\circledast)$ forms a group, then one can apply results from group theory to obtain deeper and better perspectives of the different families of fuzzy implications. For instance, it is well known that if a group G is not simple, it has a nontrivial normal subgroup N which partitions G. Now, it is easy to see that to generate the whole of G, when $O(G)<\infty$, it is sufficient to store $O(N)+O(\frac{G}{N})$ elements. Further, since any $g\in G$ is in one of these cosets, we know that $g=n\cdot g'$. If N is a nontrivial normal subgroup with some desirable properties then we have a *unique* decomposition of g into components with known properties.