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Constructing implications and coimplications on a complete lattice ☆

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Abstract

In this paper, we firstly give out the formulas for calculating the upper (lower) approximation implications and coimplications of a binary operation on a complete lattice. Then, we discuss some properties of the upper (lower) approximation implications and coimplications. Finally, we investigate the relations between the upper (lower) approximation implications and the lower (upper) approximation coimplications.

Keywords: Fuzzy connective; Implication; Coimplication; Upper (lower) approximation

1. Introduction

In fuzzy logic systems (see [13,18,25]), connectives "and", "or" and "not" are usually modeled by t-norms, t-conorms, and strong negations on $[0, 1]$ (see [1,17]), respectively. Based on these logical operators on $[0, 1]$, the three fundamental classes of fuzzy implications on $[0, 1]$, i.e., R-, S-, and QL-implications on $[0, 1]$, were defined and extensively studied (see [3–5,8,21,23]). Moreover, De Baets [9] derived coimplications from these logical operators. But, as was pointed out by Fodor and Keresztfalvi [12], sometimes there is no need of the commutativity or associativity for the connectives "and" and "or". Thus, many authors investigated implications and coimplications based on some other operators like weak t-norms [11], pseudo-t-norms [30], pseudo-uninorms [24], left and right uninorms [28,29], semi-uninorms [19], aggregation operators [22] and so on.

Constructing logical operators is an interesting topic. Recently, Jenei and Montagna [14–16] introduced several new types of constructions of left-continuous t-norms and Wang [26] laid bare the formulas for calculating the smallest pseudo-t-norm that is stronger than a binary operation and the largest implication that is weaker than a binary operation. In this paper, motivated by these works, we will further focus on this issue and investigate constructions of implications and coimplications based on binary operations.

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The paper is organized as follows. In Section 2, we briefly review the concepts of negation, N-dual operation, implication and coimplication and illustrate these concepts by means of examples. In Section 3, we give out the formulas for calculating the upper (lower) approximation implications and coimplications of binary operations. In Section 4, we discuss some properties of the upper (lower) approximation implications and coimplications. In Section 5, we investigate the relations between the upper (lower) approximation implications and the lower (upper) approximation coimplications.

The knowledge about lattices required in this paper can be found in [6].

Throughout this paper, unless otherwise stated, L always represents any given complete lattice with maximal element 1 and minimal element 0; J stands for any index set.

2. Negations, N-dual operations, implications and coimplications

In this section, we briefly review the concepts of negation, N-dual operation, implication and coimplication and illustrate these concepts by means of examples.

Definition 2.1. (See Ma and Wu [20].) A mapping $N : L \to L$ is called a negation if

(N1) $N(0) = 1$ and $N(1) = 0$,

(N2) $x \leqslant y,\ x, y \in L \Rightarrow N(y) \leqslant N(x)$.

A negation N is called strong if it is an involution, i.e., $N(N(x)) = x$ for any $x \in L$.

Example 2.1. (See De Baets [9].) Define N_W and N_M on L as follows:

$$N_W(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad N_M(x) = \begin{cases} 1, & \text{if } x \neq 1, \\ 0, & \text{otherwise}. \end{cases}$$

Then N_W and N_M are two negations. For any negation N on L, it holds that $N_W \leqslant N \leqslant N_M$.

Theorem 2.1. *(See Wang and Yu [30].) Let $x_j \in L$ $(j \in J)$. If N is a strong negation on L, then*

$$N\Big(\bigvee_{j\in J} x_j\Big) = \bigwedge_{j\in J} N(x_j), \tag{1}$$

$$N\Big(\bigwedge_{j\in J} x_j\Big) = \bigvee_{j\in J} N(x_j). \tag{2}$$

Definition 2.2. (See De Baets [9].) Consider a strong negation N on L. The N-dual operation of a binary operation A on L is the binary operation A_N on L defined by

$$A_N(x, y) = N^{-1}\big(A\big(N(x), N(y)\big)\big) \quad \forall x, y \in L. \tag{3}$$

Note that $(A_N)_{N^{-1}} = (A_N)_N = A$ for any binary operation A on L.

Definition 2.3. (See Wang and Fang [29].) A binary operation A on L is called left (right) infinitely $\vee$-distributive if

$$A\Big(\bigvee_{j\in J} x_j, y\Big) = \bigvee_{j\in J} A(x_j, y) \quad \Big(A\Big(x, \bigvee_{j\in J} y_j\Big) = \bigvee_{j\in J} A(x, y_j)\Big) \quad \forall x, y, x_j, y_j \in L;$$

left (right) infinitely $\wedge$-distributive if

$$A\Big(\bigwedge_{j\in J} x_j, y\Big) = \bigwedge_{j\in J} A(x_j, y) \quad \Big(A\Big(x, \bigwedge_{j\in J} y_j\Big) = \bigwedge_{j\in J} A(x, y_j)\Big) \quad \forall x, y, x_j, y_j \in L.$$

If a binary operation U is left infinitely $\vee$-distributive ($\wedge$-distributive) and also right infinitely $\vee$-distributive ($\wedge$-distributive), then U is said to be infinitely $\vee$-distributive ($\wedge$-distributive).