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Finitely maxitive conditional possibilities, Bayesian-like inference, disintegrability and conglomerability

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Abstract

The aim of the paper is to study Bayesian-like inference processes involving coherent finitely maxitive T-conditional possibilities assessed on infinite sets of conditional events. Coherence of an assessment consisting of an arbitrary possibilistic prior and an arbitrary possibilistic likelihood function is proved, thus a closed form expression for the envelopes of the relevant joint and posterior possibilities is given when T is the minimum or a strict t-norm. The notions of disintegrability and conglomerability are also studied and their relevance in the infinite version of the possibilistic Bayes formula is highlighted.

Keywords: Bayesian-like inference; Disintegrability; Conglomerability; Finite maxitivity; T-conditional possibility; Possibilistic likelihood function; Coherence

1. Introduction

Given a numerical or qualitative assessment on a family of (conditional) events, to make inference means to extend the given assessment to new (conditional) events, maintaining consistency with the chosen framework of reference (singled out by an uncertainty measure).

A numerical inference procedure is said Bayesian-like if the initial information consists of a prior and a likelihood assessments. The main steps of this procedure necessarily require to prove coherence of the global assessment (that is its consistence with the framework of reference) and then to compute the coherent extension on the events of interest. In case the extension is not unique, the aim is to characterize (and then compute) the upper and lower envelopes of the class of all coherent extensions.

Despite this general procedure, sometimes the bounds we obtain could be non-informative (i.e., they could reduce to 0 and 1, respectively), so we consider lower and upper envelopes of particular classes of extensions fulfilling suitable additional analytical properties, such as disintegrability or conglomerability.

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In the (finitely additive) probabilistic framework the problem of Bayesian inference has been studied in depth, starting from the seminal work [31] to the most recent one [8], where a closed form expression for the lower envelope of all the possible coherent extensions is given starting either from a finitely additive prior probability or a completely monotone prior lower probability.

The aim of this paper is to study the above problems in the framework of finitely maxitive T-conditional possibility (with T any continuous t-norm), directly introduced as a function on a suitable family of conditional events, satisfying a proper set of axioms (see [5,6,13,14]). The problems related to the updating of possibility by Bayesian-like procedures have been already studied in a finitary setting [10]: here we focus on infinite families of events. Similar problems have been considered in [17] in case of completely maxitive possibility measures and with a notion of conditioning different from the one adopted in this paper.

The paper is organized as follows. In Section 2 we give some preliminary notions on integration with respect to finitely maxitive possibilities and on coherent T-conditional possibilities on infinite sets of events and their extensions.

In Section 3, starting from a set $\mathcal{L}$ of (mutually exclusive and exhaustive) "hypotheses" and a set of (mutually exclusive and exhaustive) "evidences" $\mathcal{E}$, we focus on coherence and extensions of assessments $\{\pi, f\}$, where π is a finitely maxitive prior possibility defined on $\mathcal{A}_{\mathcal{L}}$ (Boolean algebra containing $\mathcal{L}$) and f is a possibilistic likelihood function defined on $\mathcal{A}_{\mathcal{E}} \times \mathcal{L}$ (with $\mathcal{A}_{\mathcal{E}}$ a Boolean algebra containing $\mathcal{E}$). The main results of this section state that:

- every assessment $\{\pi, f\}$ is a coherent T-conditional possibility for every continuous t-norm T;
- for $T = \min$ or a strict t-norm it is possible to characterize the lower and upper envelopes of the posterior possibilities, that is the coherent extensions of $\{\pi, f\}$ to the events in $\mathcal{A}_{\mathcal{L}} \times \mathcal{E}$.

In Section 4 we consider for arbitrary (infinite) partitions two concepts: disintegrability and conglomerability for events, defined in analogy to those introduced in probability theory [1,21,33]. As is well-known, in probability theory the two properties (see, e.g., [4,20,24,32–34]) are strictly related between them and with σ-additivity. In fact for finitely additive conditional probabilities it is possible to have examples which, contrary to intuition, show that a P need not be conglomerable (and so disintegrable). In probabilistic Bayesian literature, the phenomenon of nonconglomerability has emerged in the so-called marginalization paradoxes [7].

We note that our notion of conglomerability differs from the ones proposed for coherent lower and upper previsions (see for instance [18,29,37]), since we consider T-conditional possibilities which are not upper conditional probabilities, in general.

Finally, we show that the above notions are particularly relevant in the context of Bayesian-like inference processes. Indeed, when satisfied, they constrain the set of coherent values for the joint possibility and so for the posterior possibility.

2. Coherent T-conditional possibility

An *event* E is singled out by a Boolean proposition, that is a statement that can be either true or false. Since in general it is not known whether E is true or not, we are uncertain on E, which is said to be *possible*. Two particular events are the *certain event* Ω and the *impossible event* $\emptyset$, that coincide with, respectively, the top and the bottom of every Boolean algebra $\mathcal{B}$ of events, i.e., a set of events closed w.r.t. the familiar Boolean operations of *contrary* $(\cdot)^c$, *conjunction* $\wedge$ and *disjunction* $\vee$ and equipped with the partial order of *implication* $\subseteq$. Recall that due to Stone's theorem, events can be represented as subsets of a universe set that is identified with Ω: in this case we continue to use $(\cdot)^c$, $\wedge$ and $\vee$ in place of set-theoretic operations.

A *conditional event* $E|H$ is an ordered pair (E, H), with $H \neq \emptyset$, where E and H are events of the same "nature", but with a different role (in fact H acts as a "possible hypothesis"). In particular any event E can be seen as the conditional event $E|\Omega$.

In what follows, $\mathcal{B} \times \mathcal{H}$ denotes a set of conditional events with $\mathcal{B}$ a Boolean algebra and $\mathcal{H}$ an additive set (i.e., closed with respect to finite disjunctions) such that $\mathcal{H} \subseteq \mathcal{B}^0 = \mathcal{B} \setminus \{\emptyset\}$. Let us recall that, every set of conditional events $\mathcal{G} = \{E_i|H_i\}_{i\in I}$, can be embedded into a minimal structured set $\mathcal{B} \times \mathcal{H}$, where $\mathcal{B} = \langle\{E_i, H_i\}_{i\in I}\rangle$ is the Boolean algebra generated by the events $\{E_i, H_i\}_{i\in I}$ and $\mathcal{H} = \langle\{H_i\}_{i\in I}\rangle^{\mathbf{A}}$ is the additive set generated by $\{H_i\}_{i\in I}$.

We recall that a *t-norm* T is a commutative, associative, increasing, binary operation on $[0, 1]$, having 1 as neutral element. A t-norm is called *continuous* (analogously, *left-continuous* or *right-continuous*) if it is continuous as a