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A characterization of neo-additive measures

Bruno Girotto, Silvano Holzer *

Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche, "Bruno de Finetti", Università degli studi di Trieste, Italy

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Abstract

Neo-additive and generalized neo-additive capacities were introduced in order to capture both optimistic and pessimistic attitudes towards uncertainty without abandoning the subjective probabilistic approach. In this way, one can obtain, as particular cases, some well-known decision criteria (via Choquet expectation) adopted in Decision Theory and Mathematical Statistics.

In order to introduce these capacities, Chateauneuf, Eichberger, Grant and Eichberger, Grant, Lefort consider three types of events: universal, null and essential events; afterwards they introduce capacities which are null on null events (null property), assume value one on universal events (normalization property) and are translations of finitely additive probabilities on the family of essential events. Finally, they supply a theoretic measure characterization of these type of capacities.

In this paper, we introduce neo-additive measures as monotone measures which are translations of finitely additive ones on the family of essential events, without assumption of normalization property and null property. Moreover, we supply a simple and natural theoretic characterization of these measures obtaining, as particular cases, the corresponding results of the previous authors. In this way, our results give a robust foundation of neo-additive and generalized neo-additive capacities in abstract measure setting.

Keywords: Null; Universal and essential events; Monotone measures; Neo-additive capacities; Generalized neo-additive capacities; Neo-additive measures

1. Introduction and preliminaries

Empirical studies in decision making under risk have shown that people, facing the choice among objective lotteries, tend to overweight events with small probabilities and underweight events with large probabilities (see e.g. Gonzales and Wu [4]). One way to model such a behaviour is through a suitable inverse-S shaped weighting function w, overweighting probabilities close to zero and underweighting probabilities close to one. A simple version of this practice is to choose as w a nondecreasing affine function.

In this setting, Chateauneuf et al. in [2] introduce a suitable parametric family of normalized monotone measures. Precisely, they consider a measurable space $(\Omega, \mathcal{F})$ and a partition of the family of *events* $\mathcal{F}$ into three non-empty subsets: the family $\mathcal{U}$ of *universal events*, the family $\mathcal{N} = \{F^c : F \in \mathcal{U}\}$ of *null events* and the family $\mathcal{E} = \mathcal{F} \setminus (\mathcal{U} \cup \mathcal{N})$

* Corresponding author. Fax: +39 040 5587033.
E-mail addresses: brunog@econ.units.it (B. Girotto), holzer@units.it (S. Holzer).

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of *essential events*. Moreover, given a (finitely) additive probability π such that $\pi(\mathcal{N}) = \{0\}$ and two real numbers $\delta, \alpha \in [0, 1]$, they call *neo-additive capacity* (acronym for *non-extreme-outcome-additive capacity*) the following normalized monotone measure:

$$\nu_{\pi,\delta,\alpha}(F) = \begin{cases} 0 & \text{if } F \in \mathcal{N} \\ (1-\delta)\pi(F) + \delta\alpha & \text{if } F \in \mathcal{E} \\ 1 & \text{if } F \in \mathcal{U} \end{cases}$$
$$= (1-\delta)\pi(F) + \delta\,\nu_{\pi,1,\alpha}(F),$$

which is a mixture of the additive probability π and the *Hurwicz capacity* $\nu_{\pi,1,\alpha}$ (with degree of optimism equal to α). The interest in this type of monotone measures is that the Choquet integral of a bounded measurable function f w.r.t. $\nu_{\pi,\delta,\alpha}$ assumes the form:

$$\mathrm{C}\int_{\Omega} f d\nu_{\pi,\delta,\alpha} = (1-\delta)\,\mathrm{C}\int_{\Omega} f d\pi + \delta\left[\alpha \sup\{t : \{f > t\} \notin \mathcal{N}\} + (1-\alpha)\inf\{t : \{f > t\} \notin \mathcal{U}\}\right],$$

where $\{f > t\} = \{\omega \in \Omega : f(\omega) > t\}$. In the standard case $\mathcal{U} = \{\Omega\}$, this formula becomes:

$$\mathrm{C}\int_{\Omega} f d\nu_{\pi,\delta,\alpha} = (1-\delta)\,\mathrm{C}\int_{\Omega} f d\pi + \delta\left[\alpha \sup f + (1-\alpha)\inf f\right],$$

which shows that the Choquet integral criterion can be seen as a mixture of two well-known criteria usually adopted in decision problems under uncertainty: the Bayesian mean value criterion and the classical Hurwicz criterion. This suggests the following psychological interpretation of the parameters: δ (*degree of ambiguity*) measures the lack of confidence of the agent in the additive probability π, while α (*degree of optimism*) and $1-\alpha$ (*degree of pessimism*) represent, respectively, the weights that the ambiguous part of the agent beliefs puts on the best and on the worst possible consequence.

Moreover, we remark that several well-known decision criteria can be obtained via Choquet expectation w.r.t. a neo-additive capacity, by suitable choice of null sets and parameters δ, α; for instance:

– mean value: $\delta = 0$;
– pure pessimism: $\mathcal{N} = \{\emptyset\}$, $\delta = 1$, $\alpha = 0$;
– pure optimism: $\mathcal{N} = \{\emptyset\}$, $\delta = 1$, $\alpha = 1$;
– Hurwicz criterion: $\mathcal{N} = \{\emptyset\}$, $\delta = 1$, $0 < \alpha < 1$;
– Hodges–Lehmann criterion: $\mathcal{N} = \{\emptyset\}$, $0 < \delta < 1$, $\alpha = 0$.

Finally, Chateauneuf et al. in [2] provide a behavioral axiomatic treatment of neo-additive capacities in the context of Choquet expected utility model. To this end, they first characterize neo-additive capacities, in an abstract measure setting (see Proposition 3.1, p. 542).

Afterwards, Eichberger et al. in [3] introduce another parametric family of normalized monotone measures containing, as special cases (among others), neo-additive capacities. Given an additive probability π (non-necessarily with $\pi(\mathcal{N}) = \{0\}$) and two real numbers $b \geq 0$ and a such that $0 \leq b\pi(F) + a \leq 1$ for any $F \in \mathcal{E}$, they call *generalized neo-additive capacity* the following normalized monotone measure:

$$\eta(F) = \begin{cases} 0 & \text{if } F \in \mathcal{N} \\ b\pi(F) + a & \text{if } F \in \mathcal{E} \\ 1 & \text{if } F \in \mathcal{U}, \end{cases}$$

which is not necessarily a mixture of an additive probability and a Hurwicz capacity.

Moreover, for these monotone measures, in [3] they provide a behavioral axiomatic treatment of updating preferences in a context of dynamic Choquet expected utility model. To this end, they first characterize, in an abstract measure setting, generalized neo-additive capacities which are null-additive on $\mathcal{N}$ (see Lemma 3, p. 251). Their arguments are based on the proof of additivity of a suitable monotone measure, given by Chateauneuf et al. (see