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Short communication

Independence results for multivariate tail dependence coefficients

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Abstract

In this paper we study some properties of multivariate lower and upper tail dependence coefficients, and analyze the relationship between pairwise tail independence, mutual tail independence, and extremal independence.

Keywords: Archimedean copula; Copula; Extremal dependence; Tail dependence

1. Introduction

Tail dependence coefficients explain the asymptotic probability that all random variables in a given set become large, given that random variables of another set are also large. Extremal dependence coefficients study the asymptotic dependence structure of the minimum and the maximum of a random vector. It is well known that pairwise independence does not imply mutual independence for a finite collection of random variables. In this paper we use copulas to examine the situation for pairwise tail independence versus mutual tail independence, and its relationship with extremal independence. Understanding the tail dependence properties of a model is critically important in constructing a model for a real-world application—e.g. to understand the risk of simultaneous threshold crossing for a set of random variables. We also provide a counterexample to a statement asserted in [9] concerning the generator and the diagonal section of an Archimedean copula.

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2. Preliminaries

Let $n \geq 2$ be an integer. An n-dimensional *copula* (*n-copula*, for short) is a multivariate distribution function whose univariate margins are uniformly distributed on $\mathbb{I}(=[0,1])$. The importance of copulas in statistics is described in Sklar's theorem [21]: Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random n-vector with multivariate distribution function H and respective univariate marginal distribution functions $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C (which is uniquely determined on $\times_{i=1}^n \text{Range}\, F_i$) such that the following equality holds:

$$H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \quad \forall \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \overline{\mathbb{R}}^n \left(= [-\infty, \infty]^n\right).$$

Thus copulas link joint distribution functions to their univariate margins. If the random vector $\mathbf{X}$ is continuous, then we can write

$$C(\mathbf{u}) = H\left(F_1^{(-1)}(u_1), F_2^{(-1)}(u_2), \ldots, F_n^{(-1)}(u_n)\right), \quad \forall \mathbf{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{I}^n,$$

where C is the unique n-copula associated with $\mathbf{X}$, and $F_i^{(-1)}$ is the *quasi-inverse* of F_i, i.e., $F_i^{(-1)}(t) = \inf\{x \in \overline{\mathbb{R}}: F_i(x) \geq t\}$. For a complete survey on copulas and their applications in several fields, see, for instance, [2,3,13,14,19,20].

A vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ of random variables is (mutually) *n-exchangeable* [15] if for any permutation σ of $\{1, 2, \ldots, n\}$, the vector $\left(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)}\right)$ has the same distribution as $\mathbf{X}$. Consequently, if the elements of $\mathbf{X}$ are continuous and identically distributed—with associated copula C—, pairwise and mutual exchangeability of $\mathbf{X}$ correspond, respectively, to symmetry of the 2-margins of the copula C and *permutation symmetry* of C, i.e., $C(\mathbf{u}) = C\left(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(n)}\right)$ for any permutation σ of $\{1, 2, \ldots, n\}$.

For any random variables $X_1, X_2, \ldots, X_n$ with respective distribution functions $F_1, F_2, \ldots, F_n$ and associated n-copula C, let $\widehat{C}$ denote the *survival copula* of C, which is given by

$$\begin{aligned}\widehat{C}(\mathbf{u}) &= \mathbb{P}[F_1(X_1) > 1 - u_1, \ldots, F_n(X_n) > 1 - u_n] \\ &= \sum_{i=1}^n u_i - n + 1 + \sum_{h=2}^n (-1)^h \sum_{1 \leq i_1 < \cdots < i_h \leq n} C_{i_1 \cdots i_h}(1 - u_{i_1}, \ldots, 1 - u_{i_h})\end{aligned} \tag{1}$$

for every $\mathbf{u} \in \mathbb{I}^n$, where $C_{i_1 \cdots i_h}$ is the marginal of C given by

$$C_{i_1 \cdots i_h}(u_{i_1}, u_{i_2}, \ldots, u_{i_h}) = C(1, \ldots, 1, u_{i_1}, 1, \ldots, 1, u_{i_2}, 1, \ldots, 1, u_{i_h}, 1, \ldots, 1).$$

The *diagonal section* of an n-copula C, denoted by $\delta_C(t)$, is defined by $\delta_C(t) = C(t, \ldots, t)$ for every $t \in \mathbb{I}$. Additionally, $\delta_{i_1 \cdots i_h}$ will denote the function $\delta_{i_1 \cdots i_h}(t) = C_{i_1 \cdots i_h}(t, \ldots, t)$—being $\delta_{12 \cdots n}(t) = \delta_C(t)$—, for $t \in \mathbb{I}$; and $\delta_i(t) = t$ for $i \in \{1, 2, \ldots, n\}$. For $n = 2$, it is known that δ_C can be used to study the *tail dependence* of a continuous pair of random variables (X, Y) with respective marginal distribution functions F and G [15,19]: The *lower* and *upper tail dependence parameters*, λ_L and λ_U, which are defined as

$$\lambda_L = \lim_{t \to 0^+} \mathbb{P}\left[Y \leq G^{(-1)}(t) | X \leq F^{(-1)}(t)\right] \quad \text{and} \quad \lambda_U = \lim_{t \to 1^-} \mathbb{P}\left[Y > G^{(-1)}(t) | X > F^{(-1)}(t)\right]$$

(if the limits exist) can be computed as follows: $\lambda_U = 2 - \delta_C'(1^-)$ and $\lambda_L = \delta_C'(0^+)$. Tail dependence has been shown to be useful to describe the dependence between risks in financial and actuarial risk management (see, for instance, [7,8]).

3. Multivariate tail dependence coefficients

The concepts of multivariate tail dependence coefficients are given in the following definition.

Definition 1. (See [5,16].) Let $X_1, X_2, \ldots, X_n$ be n continuous random variables with respective distribution functions $F_1, F_2, \ldots, F_n$. Let $I = \{1, 2, \ldots, n\}$ and consider two non-empty subsets $I_h \subset I$ and $J_h = I \backslash I_h$ with respective cardinal $h \geq 1$ and $n - h \geq 1$. The *multivariate lower and upper tail dependence coefficients* are given by