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Short Communication

Baire category results for exchangeable copulas

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Abstract

Considering two different metrics on the space of two-dimensional copulas $\mathscr{C}$ we prove some Baire category results for important subclasses of copulas, including the families of exchangeable, associative, and Archimedean copulas. From the point of view of Baire categories, with respect to the uniform metric d_∞, a typical copula is not symmetric and a typical symmetric copula is not associative, whereas a typical associative copula is Archimedean and a typical Archimedean copula is strict. The results in particular answer the open question posed in [1] whether the family of associative copulas is of first category in $(\mathscr{C}, d_\infty)$.

Keywords: Copula; Archimedean copula; Associativity; Exchangeability; Baire category

1. Introduction

Copulas are (the restriction to $[0,1]^d$ of) distribution functions of probability measures on $[0,1]^d$ $(d \geq 2)$ whose one-dimensional marginals are uniformly distributed on $[0,1]$. Considering that, according to Sklar's theorem [6,7,16,17], every distribution function of a random vector can be expressed as composition of a suitable copula and the corresponding marginal distribution functions, copulas are the natural building blocks of modern multivariate analysis. Having a variety of copulas at one's disposal may help in building different stochastic models that possibly differ in features being of essential importance in applications (e.g. tail behavior). Nevertheless, taking into account numerical and analytic aspects, in practice copulas are chosen from few well-studied standard (parametric or semi-parametric) families and it is arguable whether they actually are "small" or "large" families of copulas.

In order to characterize the relative size of subclasses of copulas, we will use a topological approach (as also suggested in [15]) and work with Baire categories (see, e.g. [14]). Doing so we consider the topologies induced by two different metrics on the space $\mathscr{C}$ of two-dimensional copulas: the standard uniform metric d_∞ as well as the stronger metric D_1 introduced and studied in [9,18]. In both cases the resulting metric spaces are complete (in case of

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d_∞ even compact). A subset N of a metric space (Ω, d) is called *nowhere dense* if it is not dense in any non-degenerate open ball $B(x, r)$ of radius $r > 0$ (equivalently, if its closure has empty interior). A set $A \subseteq \Omega$ is called *meager* or of *first category* in (Ω, d) if it can be expressed as (or covered by) a countable union of nowhere dense sets. A is called of *second category* if it is not meager. Finally, A is called *co-meager* (or residual) if $A^c = \Omega \setminus A$ is meager. Following [3], in complete metric spaces, first category sets are the "small" sets, co-meager sets are the "large" sets and second-category sets are merely "not small". Loosely speaking (and following conventions in the literature), we will therefore refer to the elements of a co-meager set as (topologically) *typical* and to the elements of a meager set as (topologically) *atypical* in Ω.

In the sequel $\mathscr{C}_e$ will denote the family of all exchangeable (i.e. symmetric) copulas, $\mathscr{C}_a$ the family of all associative copulas, $\mathscr{C}_{ar}$ the family of all Archimedean copulas, and $\mathscr{C}_{ar}^s$ the family of all strict Archimedean copulas, i.e. Archimedean copulas whose generator φ fulfills $\varphi(0) = \infty$ (see [13]). It is well known that we have $\mathscr{C}_{ar} \subset \mathscr{C}_a \subset \mathscr{C}_e$ and that $\mathscr{C}_a$ and $\mathscr{C}_e$ are closed in $(\mathscr{C}, d_\infty)$ (see, for instance, [10]). Since convergence w.r.t. D_1 implies convergence w.r.t. d_∞ (see again [18]) the families $\mathscr{C}_a$ and $\mathscr{C}_e$ are also closed in $(\mathscr{C}, D_1)$.

We will prove the following results:

- The family of exchangeable copulas $\mathscr{C}_e$ is nowhere dense (hence of first category) in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$.
- The family $\mathscr{C}_a$ of associative copulas is nowhere dense in $(\mathscr{C}_e, d_\infty)$ as well as in $(\mathscr{C}_e, D_1)$.
- The family $\mathscr{C}_{ar}$ of Archimedean copulas is co-meager (hence of second category) in $(\mathscr{C}_a, d_\infty)$.
- The family $\mathscr{C}_{ar}^s$ of all strict Archimedean copulas is co-meager in $(\mathscr{C}_{ar}, d_\infty)$.

As a byproduct, we give an affirmative answer to the open problem posed in [1, Problem 10] (also see [2]) asking whether the family of associative copulas is of first category in $(\mathscr{C}, d_\infty)$.

Remark 1.1. The results presented in this paper are not intended to suggest any families of copulas to be used in practice but merely to give a purely topological characterization of some well-known classes. In fact, as pointed out before, (in complete metric spaces) Baire categories establish a rough classification of subsets as "small", "large", or "not small" but do not allow for a more accurate quantification of size, implying that Baire categories are not useful for deciding which parametric classes of copulas should (or should not) be used in practice.

2. The results

For basic definitions and properties of copulas we refer to [8,13], for the metric D_1 to [18], and directly start with the following result.

Theorem 2.1. *$\mathscr{C}_e$ is nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$.*

Proof. For every $A \in \mathscr{C}$ there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of non-exchangeable copulas that converges to A w.r.t. D_1 (hence w.r.t. d_∞). In fact, if A itself is not exchangeable we may choose $A_n = A$ and if A is exchangeable we may consider $A_n := (1 - 1/n)A + (1/n)E$ with $E \in \mathscr{C}$ being an arbitrary asymmetric copula. As immediate consequence $\mathscr{C}_e$ cannot contain any nonempty open subset of $(\mathscr{C}, D_1)$ or of $(\mathscr{C}, d_\infty)$. □

Having Theorem 2.1 we immediately get that each subclass of $\mathscr{C}_e$ (including the family of all Gaussian copulas, Gumbel copulas, Clayton copulas, symmetric Bernstein copulas, symmetric checkerboard copulas, and many more) is nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$ too. Additionally, considering that, according to [5,19], all idempotent copulas (idempotent with respect to the star product introduced in [4]) are symmetric, and, letting $\mathscr{C}_i$ the family of all idempotent copulas, we also get that $\mathscr{C}_i$ is nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$. The subsequent corollary serves to gather some important consequences – in particular it gives a positive answer to Problem 10 in [1] asking whether the family of associative copulas is of first category in $(\mathscr{C}, d_\infty)$.

Corollary 2.2. *$\mathscr{C}_{ar}$, $\mathscr{C}_a$ and $\mathscr{C}_i$ are nowhere dense in $(\mathscr{C}, d_\infty)$ and in $(\mathscr{C}, D_1)$. In particular, all three families are of first category in $(\mathscr{C}, d_\infty)$ as well as in $(\mathscr{C}, D_1)$.*