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# On some questions concerning the axiomatisation of WNM-algebras and their subvarieties

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**Abstract**

In a seminal paper Esteva and Godo introduced monoidal $t$-norm-based logic MTL and some of its prominent extensions such as NM and WNM. We notice that NM is axiomatisable from IMTL, and hence MTL, with one-variable axioms, by instantiating the WNM axiom over one variable. This observation leads us here to study the logic axiomatised by extending MTL by this one-variable axiom. We shall refer to its equivalent algebraic semantics as the variety of GHP-algebras, for those algebras will be shown to form the largest variety of MTL-algebras such that the *falsum*-free reducts of the positive cones of their chains are the most general totally ordered Gödel hoops. Among other results we obtain a general description of GHP standard algebras, and use the latter to characterise those extensions of WNM that can be obtained from GHP via the same set of extending axioms.


*Keywords:* Many-valued logics; MTL; Weak Nilpotent Minimum; Nilpotent Minimum; $t$-Norms; Axiomatisation of varieties

## 1. Introduction

The fundamental paper [24] by Esteva and Godo introduces and studies the logic MTL, or *monoidal $t$-norm-based logic*, which has been successively shown in [36] to be *the* logic of all left-continuous $t$-norm and their residua. In that paper the authors introduce some extensions of MTL and prove some of their properties. In particular they introduce the logic NM based on the nilpotent minimum $t$-norm, and its generalisation WNM.

The logic WNM of *weak nilpotent minimum* is axiomatised adding the axiom

$$\neg(\varphi\&\psi) \vee \big((\varphi \wedge \psi) \to (\varphi\&\psi)\big), \qquad \text{(WNM)}$$

to MTL. WNM plays a relevant role in the hierarchy of extensions of MTL. As a matter of fact, among the extensions of WNM we find fundamental t-norm based logics as Gödel logics, and the prototypical example of a logic based on a left-continuous but not continuous $t$-norm, namely NM [30]. The $t$-norm based logic RDP [35] of revised drastic

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product, and the logic DP [3] (or, $S_3$MTL [34,39]) of drastic product chains are both extensions of WNM (recall that the drastic product $t$-norm, being the smallest possible, is considered one of the fundamental $t$-norms, but, as it is not left-continuous, it is not the basis of any $t$-norm logic, for it lacks the residuum). Moreover, the hierarchy of extensions of WNM contains one of the first studied examples of a logic based on a $t$-norm which is an ordinal sum that lies outside the BL hierarchy (namely, the logic NMG [41], based on the ordinal sum of NM and Gödel $t$-norms), and the first studied example of a logic based on an ordinal sum of subnorms (namely, the logic RDP, based on the ordinal sum of the subnorm constantly 0 and of Gödel $t$-norm).

Several theoretical applications, constructions, and developments in the framework of MTL logics are often exemplified on WNM or some of its prominent extensions. Undoubtedly, Gödel logic, that is intuitionistic propositional logic with prelinearity, is the most studied extension of WNM. Nevertheless, several other extensions have been subjected to investigations in the last few years. To name just a few examples, states and connection with probability theory as studied for NM in [7]; connections with others non-classical logics, for instance Nelson's constructive logic, are dealt with in [18]; alternative, temporal semantics, are explored in [13]; extensions with truth constants, as well as the first-order case have been subjected to investigations [12,26–29], too. The paper [31] classifies all the subvarieties of nilpotent minimum algebras. The PhD dissertation of Carles Noguera [39] contains one chapter entirely devoted to the study of WNM.

It is easy to see that the equivalent algebraic semantics of WNM, the variety $\mathbb{WNM}$ of WNM-algebras, is locally finite, and this fact allows to study WNM and its extensions via combinatorial means. The papers [4,6,8,9,17] present several results concerning combinatorial representation and combinatorially defined categories dually equivalent to subvarieties of $\mathbb{WNM}$.

Local finiteness is due to the structure of WNM-chains, which are easily described as follows:

$$x * y = \begin{cases} 0 & \text{if } x \leq \sim y \\ \min\{x, y\} & \text{otherwise.} \end{cases} \qquad x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ \max\{\sim x, y\} & \text{otherwise.} \end{cases} \tag{2}$$

It is worth observing that the choice of a weak negation function on a totally ordered set completely determines one WNM-chain structure, and each WNM-chain arises in this way.

On the other hand, the fact that operations on a WNM-chain are given by Eq. (2), is reflected quite transparently in the reading of axiom (WNM), which can be paraphrased as:

"*for each two elements of a chain, either their monoidal conjunction is null, or it coincides with the lattice one*".

This degree of clarity and succinctness in the axiomatisation of WNM from MTL is not achieved in many of the available axiomatisations of the relevant schematic extensions of WNM quoted above. It is precisely the need to defend this point of view that has been the first motivation for this contribution. As a matter of fact the first result we discuss is that NM can be axiomatised from IMTL (and hence, from MTL as well) using just one-variable axioms,[1] replacing the two-variable axiom (WNM) by its instantiation (WNM1) over one variable only:

$$\neg(\varphi \& \varphi) \vee \big(\varphi \to (\varphi \& \varphi)\big). \tag{WNM1}$$

This axiom can be read as:

"*for each element of a chain, either its monoidal square is null, or it coincides with the identity*".

We study the subvariety of $\mathbb{MTL}$ corresponding to the extension of MTL by the axiom (WNM1) and we obtain a general description of the chains in this variety, which we shall call $\mathbb{GHP}$, since we shall show that they form the largest subvariety of $\mathbb{MTL}$ such that the *falsum*-free reducts of the positive cones of its chains are exactly the chains in the variety of Gödel hoops (see [1,25], for backgrounds on hoops). We shall then generalise the result obtained for NM, characterising those extensions of WNM that can be obtained as well as extensions of GHP using the same set of extending axioms (if the latter preserve negation, in a technical sense that we shall explain).

[1] As one of the reviewers pointed out, this result has been already shown in [43]. That paper is in Chinese and we were not aware of its existence. Both proofs we give in this paper differ from the proof in [43]. Corollary 3.1 is based on the notion of rotational invariance of IMTL chains, while Theorem 7.1 is a straightforward application of the notions of negation-preserving equations and of D-stability that we shall introduce in Section 6.