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Paraconsistent fuzzy logic preserving non-falsity

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Abstract

We introduce proof systems and semantics for two paraconsistent extensions of the system **T** of Anderson and Belnap, and prove strong soundness, completeness, and decidability for both. The semantics of both systems is based on excluding just one element from the set of designated values. One of the systems has the variable sharing property, and so it is a relevant logic. The other is an extension of the first that may be viewed as a semi-relevant counterpart of Łukasiewicz Logic which preserves non-falsity rather than truth.
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1. Introduction

Paraconsistent logics are logics which allow non-trivial inconsistent theories. In other words: unlike in classical logic, in paraconsistent logics a single contradiction does not necessarily imply everything. Fuzzy logics, on the other hand, are logics which are based on the idea of *degrees of truth*, according to which the truth-value assigned to a proposition that involves imprecise concepts (like "tall" or "old") might not be one of the two classical values 0 and 1, but any real number between them.

Now none of the standard fuzzy logics investigated in the literature (see [5] for extensive surveys) is paraconsistent. The reason is that their consequence relation is based on preserving absolute truth (i.e. $T \vdash \varphi$ iff every legal valuation that assigns 1 to all elements of T assigns 1 to φ as well).[1] In order to develop useful paraconsistent fuzzy logics it is necessary to replace this consequence relation of the standard fuzzy logics by a less strict one, and the obvious way to do so is to use as the set of designated values a set which is more comprehensive than just $\{1\}$.

The main goal of this paper is to present a paraconsistent counterpart (called **FT**) of Łukasiewicz Logic $\mathbf{Ł}_\infty$ which reflects the above idea. **FT** has the same set of basic connectives ($\{\neg, \vee, \wedge, \rightarrow\}$) as $\mathbf{Ł}_\infty$, and like $\mathbf{Ł}_\infty$ the semantics of **FT** is based on taking the real unit interval $[0, 1]$ as the set of truth-values. Both logics also have there the same interpretations of $\wedge$, $\vee$, and (most importantly) $\neg$. Moreover, like in $\mathbf{Ł}_\infty$ (and other standard fuzzy logics), the main

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[1] Recently degree-preserving fuzzy logics which *are* paraconsistent were investigated in [7]. We shall return to this issue in the final section of this paper.

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function of $\to$ in **FT** is to make it possible to use the set $\mathcal{D}$ of the designated values for characterizing the order relation $\leq$ of the truth values: $a \leq b$ iff $a \to b \in \mathcal{D}$ (where we denote the interpretation of $\to$ again by $\to$). However, while $\mathbf{Ł}_\infty$ has a single designated value: 1, **FT** has a single *non-designated* value: 0. In other words: while the standard fuzzy logics preserve truth, **FT** preserves non-falsity. [2]

Another important feature of **FT** is that it belongs to Anderson and Belnap's family of relevant and semi-relevant logics, since it is obtained by extending Anderson and Belnap's favorite system **E** (or just the weaker system **T**) with three axioms schemas (two of which are valid in all fuzzy logics ever studied, while the third reflects our very liberal choice of the set of designated values). Now **FT** itself cannot be taken as a relevant logic, since it does not have the variable-sharing property. However, it can be viewed as a semi-relevant system, since it satisfies the same criterion of semi-relevance as the well-known semi-relevant system **RM** (see Proposition 6.7). Another important property that **FT** shares with **RM** (while most strictly relevant systems lack it) is its being *decidable*.

On our way to introduce and investigate **FT** we introduce and investigate a weaker (but still decidable) system, **TMP**, which has an interest of its own. The semantics of **TMP** is similar to that of **FT** in being based on the idea of ordered truth-degrees. However, in **TMP** the order relation of the truth-degrees is not demanded to be linear. As a result, **TMP** does have the variable-sharing property, and so it can be viewed as a relevant logic.

2. The logics TMP and FT

Notations and conventions We denote by $\mathcal{L}_\mathbf{T}$ the propositional language $\{\to, \neg, \wedge, \vee\}$, and by $\mathcal{F}_{\mathcal{L}_\mathbf{T}}$ the set of formulas of $\mathcal{L}_\mathbf{T}$. $\varphi, \psi, \sigma, \theta$ will vary over the elements of $\mathcal{F}_{\mathcal{L}_\mathbf{T}}$, and p, q, P, Q will vary over the atomic formulas of $\mathcal{L}_\mathbf{T}$.

We start by recalling the most famous relevant and semi-relevant logics in $\mathcal{L}_\mathbf{T}$ (see [1,6,3]):

Definition 2.1.

1. The logic **T** is defined by the Hilbert-type system given in Fig. 1.
2. The system **E** is obtained by adding to **T** the following axiom:

   [Esa] $\big((\varphi \to \varphi) \wedge (\psi \to \psi) \to \theta\big) \to \theta$

3. The system **R** is obtained by adding to **T** (or to **E**) the following axiom:

   [Pe] $\varphi \to \big((\varphi \to \psi) \to \psi\big)$ (Permutation)

4. The system **RM** is obtained by adding to **R** the following axiom:

   [Mi] $\varphi \to (\varphi \to \varphi)$ (Mingle)

The following proposition provides two important properties that every axiomatic extension of **T** in its language (in particular **T**, **E**, **R**, and **RM**) enjoys. For proofs see [1] or [3].

Proposition 2.2. *Let the logic* **L** *be obtained from* **T** *by the addition of some axiom schemas in* $\mathcal{L}_\mathbf{T}$.

1. *The rule of substitution of equivalents is derivable in* **L***:*

   $\varphi \leftrightarrow \psi \vdash_\mathbf{L} \theta\{\varphi/p\} \leftrightarrow \theta\{\psi/p\}$

   (where $\theta\{\varphi/p\}$ denotes the substitution in θ of φ for the atomic variable p, and $\varphi \leftrightarrow \psi =_{Df} (\varphi \to \psi) \wedge (\psi \to \varphi)$).
2. *$T, \varphi \vee \psi \vdash_\mathbf{L} \theta$ iff both $T, \varphi \vdash_\mathbf{L} \theta$ and $T, \psi \vdash_\mathbf{L} \theta$.*

Next we turn to the close relatives of the above systems which will interest us in this paper:

[2] Another way in which **FT** is a sort of dual to $\mathbf{Ł}_\infty$ is in the way it relates to the basic structural rules of Gentzen: while $\mathbf{Ł}_\infty$ accepts the implicational axioms which correspond to the weakening rule and the permutation (or exchange) rule, but reject the one that corresponds to contraction, **FT** accepts the latter axiom but rejects the former two.

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