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Fuzzy Sets and Systems 292 (2016) 130–149

FUZZY sets and systems

www.elsevier.com/locate/fss

Proof search and Co-NP completeness for many-valued logics

Mattia Bongini [a], Agata Ciabattoni [b,*], Franco Montagna [c]

[a] *Department of Mathematics, Munich University of Technology, Germany*
[b] *Department of Computer Languages, Vienna University of Technology, Austria*
[c] *Department of Information Engineering and Mathematics, University of Siena, Italy*

Received 7 February 2014; received in revised form 12 January 2015; accepted 26 February 2015

Available online 5 March 2015

Abstract

We provide a methodology to introduce proof search oriented calculi for a large class of many-valued logics, and a sufficient condition for their Co-NP completeness. Our results apply to many well known logics including Gödel, Łukasiewicz and Product Logic, as well as Hájek's Basic Fuzzy Logic.

1. Introduction

The invertibility of rules[1] in a proof system is an important feature for guiding proof search; in addition it turns out to be very useful to settle the computational complexity of the formalized logic. For many-valued logics, calculi with invertible rules (*proof search oriented calculi*) have been provided for all finite-valued logics. These calculi are defined by generalizing Gentzen sequents $A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$ to many placed (or labeled) sequents, each corresponding to a truth value of the logic, see e.g. the survey [9] ([12], for the non-deterministic case). The construction of these calculi, out of the truth tables of the connectives, is even computerized, see [10]. This design does not apply to infinite-valued logics where, excepting Gödel logic [8,5], proof search oriented calculi – when available – are introduced on a logic by logic basis and their construction requires some ingenuity; this is for instance the case of the calculi for Łukasiewicz and Product logic [29,28,27], defined using *hypersequents*, which are finite "disjunctions" of Gentzen sequents [4,3].

An important step towards the automated construction of proof search oriented calculi for many-valued logics was taken in [8] with the introduction of *sequents of relations*, that are disjunctions of semantic predicates over formulas, and of a methodology to construct such calculi for all *projective* logics. Intuitively a logic is projective if for each connective $\square$, the value of $\square(x_1, \ldots, x_n)$ is equal to a constant or to one of the $x_1, \ldots, x_n$. The methodology was extended in [19] to handle *semi-projective logics* where the value of each $\square(x_1, \ldots, x_n)$ can also be a term of the form

* Corresponding author.
E-mail addresses: mattia.bongini@ma.tum.de (M. Bongini), agata@logic.at (A. Ciabattoni), montagna@unisi.it (F. Montagna).

[1] The premises are derivable whenever their conclusions are.

http://dx.doi.org/10.1016/j.fss.2015.02.016
0165-0114/

$p(x_i)$ with p unary function symbol and $i \in \{1, \ldots, n\}$. Projective logics are quite interesting, but perhaps not general enough: among many-valued logics, only the finite-valued logics and Gödel logic are projective. Semi-projective logics constitute a slightly larger class, and they capture, for instance, Nilpotent and Weak Nilpotent Minimum logic [21], the relevance logic *RM* [2] and, by considering conservative extensions, Hájek's Basic Fuzzy Logic BL extended with n-contraction [14]. All semi-projective logics have a locally finite variety as their equivalent algebraic semantics while important many-valued logics such as Łukasiewicz logic, Product Logic and BL do not, despite the fact that they have suitable calculi with invertible rules [18,30,16] and are Co-NP complete. The calculi in [18,30,16] are defined using relational hypersequents (*r-hypersequents* for short) that generalize hypersequents by considering finite disjunctions of two different types of sequents, where Gentzen's sequent arrow $\Rightarrow$ is replaced in one by $<$ and in the other by $\leq$.

In this paper we generalize r-hypersequents to disjunctions of arbitrary semantic predicates (not only $<$ and $\leq$) over *multisets* of formulas, rather than single formulas as in the case of sequents of relations. We introduce a methodology to define r-hypersequent calculi for a large class of many-valued logics (*hyperprojective logics*) and identify sufficient conditions on these calculi that guarantee the Co-NP completeness of the formalized logics. Our methodology applies to projective and semi-projective logics as well as to Łukasiewicz, Product Logic and BL; it subsumes all existing results on sequent of relations and on r-hypersequent calculi (e.g. [8,6,19,18,30,16]), and provides a unified perspective on most of the known complexity results for many-valued logics. Moreover, our method can be applied to new logics (or already known logics not having yet proof search oriented calculi), provided that they are hyperprojective.

In a hyperprojective logic the value of each connective $\Box(x_1, \ldots, x_n)$ is defined by cases this time expressed by relations on *multisets* of constants, of terms $x_1, \ldots, x_n$ and of $p(x_i)$, for p unary functions and $i \in \{1, \ldots, n\}$.

We illustrate the idea behind hyperprojective logics and the way we define r-hypersequent calculi for them with the example of Product Logic. For this logic, as in (the projective presentation of) Gödel Logic [8,6,7] it is natural to consider the relations $<$ and $\leq$. Product Logic is neither projective nor semi-projective, because if $x, y \notin \{0, 1\}$ then the product $x\&y$ depends on *both* x and y. The idea is to represent the product by a monoidal operation $\oplus$ standing for the union of multisets, i.e. $x\&y = x \oplus y$.

In general, to define invertible rules for a connective $\Box(x_1, \ldots, x_n)$ of a hyperprojective logic L we will consider "reductions" (based on the relations in the semantic theory of L) that act on multisets of formulas, i.e., on $\Gamma \oplus \Box(x_1, \ldots, x_n)$, where Γ is a multiset, and in which the formula $\Box(x_1, \ldots, x_n)$ is decomposed into a *multiset* of smaller terms (constants, x_i or $p(x_i)$).

In the particular case of the connective $x\&y$ of Product Logic we consider the following "reduction cases": for $\Gamma \oplus x\&y \lhd \Delta$ as $\Gamma \oplus x \oplus y \lhd \Delta$ and for $\Gamma \lhd \Delta \oplus x\&y$ as $\Gamma \lhd \Delta \oplus x \oplus y$, where $\lhd$ denotes either $<$ or $\leq$. Our calculus for Product Logic will then contain r-hypersequents consisting of disjunctions of sequents of the form $\Gamma < \Delta$ or $\Gamma \leq \Delta$, where Γ and Δ are multisets of formulas. As in the case of hypersequents [4,3] the disjunction will be denoted by "|" and the union of multisets by ",". With this notation we have that $\phi\&\psi, \Gamma \lhd \Delta$ reduces (and it is indeed equivalent to) to $\phi, \psi, \Gamma \lhd \Delta$ and $\Gamma \lhd \Delta, \phi\&\psi$ reduces to $\Gamma \lhd \Delta, \phi, \psi$, which naturally lead to the following left and right rules for the connective & w.r.t. the relation $\lhd$ (below H stands for an arbitrary r-hypersequent)

$$\frac{H|\phi, \psi, \Gamma \lhd \Delta}{H|\phi\&\psi, \Gamma \lhd \Delta} \qquad \frac{H|\Gamma \lhd \Delta, \phi, \psi}{H|\Gamma \lhd \Delta, \phi\&\psi}$$

Now since

$$x \wedge y = \begin{cases} x & \text{if } x \leq y \\ y & \text{if } y < x \end{cases} \qquad x \vee y = \begin{cases} y & \text{if } x \leq y \\ x & \text{if } y < x \end{cases}$$

we have:

- $\Gamma, \phi \vee \psi \lhd \Delta$ reduces to $\psi < \phi|\Gamma, \psi \lhd \Delta$ and to $\psi \leq \phi|\Gamma, \phi \lhd \Delta$;
- $\Gamma \lhd \Delta, \phi \vee \psi$ reduces to $\psi < \phi|\Gamma \lhd \Delta, \psi$ and to $\phi \leq \psi|\Gamma \lhd \Delta, \phi$;
- $\Gamma, \phi \wedge \psi \lhd \Delta$ reduces to $\psi < \phi|\Gamma, \phi \lhd \Delta$ and to $\phi \leq \psi|\Gamma, \psi \lhd \Delta$;
- $\Gamma \lhd \Delta, \phi \wedge \psi$ reduces to $\psi < \phi|\Gamma \lhd \Delta, \phi$ and to $\phi \leq \psi|\Gamma \lhd \Delta, \psi$.

Finally, recalling that in Product Logic

$$x \to y = \begin{cases} \frac{y}{x} & \text{if } y < x \\ 1 & \text{if } x \leq y \end{cases}$$