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Polyhedral MV-algebras

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Dedicated to Francesc Esteva, on his 70th birthday

Abstract

A *polyhedron* in $\mathbb{R}^n$ is a finite union of simplexes in $\mathbb{R}^n$. An MV-algebra is *polyhedral* if it is isomorphic to the MV-algebra of all continuous $[0, 1]$-valued piecewise linear functions with integer coefficients, defined on some polyhedron P in $\mathbb{R}^n$. We characterize polyhedral MV-algebras as finitely generated subalgebras of semisimple tensor products $S \otimes F$ with S simple and F finitely presented. We establish a duality between the category of polyhedral MV-algebras and the category of polyhedra with $\mathbb{Z}$-maps. We prove that polyhedral MV-algebras are preserved under various kinds of operations, and have the amalgamation property. Strengthening the Hay–Wójcicki theorem, we prove that every polyhedral MV-algebra is strongly semisimple, in the sense of Dubuc–Poveda.

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1. Introduction and preliminary material

This paper is devoted to polyhedral MV-algebras. On the one hand, these algebras constitute a proper subclass of finitely generated strongly semisimple MV-algebras, and are a generalization of finitely presented MV-algebras. On the other hand, polyhedral MV-algebras with homomorphisms are dual to polyhedra in euclidean space, *equipped with $\mathbb{Z}$-maps* (Definition 3.1). $\mathbb{Z}$-homeomorphism of two polyhedra $P, Q \subseteq \mathbb{R}^n$ amounts to their continuous $\mathcal{G}_n$-equidissectability, where $\mathcal{G}_n = \mathsf{GL}(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$ is the n-dimensional affine group over the integers, [18]. In the resulting new geometry, already rational polyhedra, with their wealth of combinatorial and numerical invariants, pose challenging algebraic-topological, measure-theoretic and algorithmic problems, [4–6].

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Our paper is organized as follows: Section 2 is devoted to proving the characterization of polyhedral MV-algebras as finitely generated subalgebras of semisimple tensor products $S \otimes F$, with S simple and F finitely presented. In Section 3 we give a virtually self-contained proof of the duality between the category of polyhedral MV-algebras and the category of polyhedra with $\mathbb{Z}$-maps. In Section 4 we prove that polyhedral MV-algebras have the amalgamation property. In Section 5 it is shown that polyhedral MV-algebras are strongly semisimple, in the sense of Dubuc–Poveda [8]. This generalizes the Hay–Wójcicki theorem [10,20].

We refer to [11] and [19] for background on polyhedral topology. A set $Q \subseteq \mathbb{R}^n$ is said to be a *polyhedron* if it is a finite union of simplexes $S_i \subseteq \mathbb{R}^n$. Thus Q need not be convex, nor connected; the simplexes S_i need not have the same dimension. If each S_i can be chosen with rational vertices, then Q is said to be a *rational polyhedron*.

For any integer $n, m > 0$ and polyhedron $P \subseteq \mathbb{R}^n$, a function $f : P \to \mathbb{R}^m$ is *piecewise linear* if it is continuous and there are finitely many linear transformations $L_1, \dots, L_u : \mathbb{R}^n \to \mathbb{R}^m$ such that for each $x \in P$ there is an index $i \in \{1, \dots, u\}$ with $f(x) = L_i(x)$. The adjective "linear" is always understood in the affine sense. If in particular the coefficients of $L_1, \dots, L_u$ are integers, we say that f is *piecewise linear with integer coefficients*.

We refer to [7] and [17] for background on MV-algebras. For any polyhedron $P \subseteq \mathbb{R}^n$ we let $\mathcal{M}(P)$ denote the MV-algebra of piecewise linear functions $f : P \to [0, 1]$ with integer coefficients and the pointwise operations of negation $\neg x = 1 - x$ and truncated addition $x \oplus y = \min(1, x + y)$. By [7, 3.6.7], $\mathcal{M}(P)$ is a semisimple MV-algebra. $\mathcal{M}([0, 1]^n)$ is the *free n-generator MV-algebra*. This is McNaughton's theorem, [7, 9.1.5]. By [17, 6.3], an MV-algebra A is finitely presented iff it is isomorphic to $\mathcal{M}(R)$ for some rational polyhedron $R \subseteq [0, 1]^n$. An MV-algebra A is said to be *polyhedral* if, for some $n = 1, 2, \dots$, it is isomorphic to $\mathcal{M}(P)$ for some polyhedron $P \subseteq \mathbb{R}^n$.

Unless otherwise specified, all polyhedra in this paper are nonempty, and all MV-algebras are nontrivial.

2. A characterization of polyhedral MV-algebras

Lemma 2.1. *For any polyhedron $P \subseteq \mathbb{R}^n$ and function $f : P \to [0, 1]$, the following conditions are equivalent:*

(i) *f is piecewise linear. (As specified in the first lines of Section 1, piecewise linearity entails continuity.)*
(ii) *For some triangulation Δ of P, f is linear on each simplex of Δ.*
(iii) *For any cube $C = [a, b]^n \subseteq \mathbb{R}^n$ containing P there is a piecewise linear function $g : C \to [0, 1]$ such that f is the restriction of g to P, in symbols, $f = g \restriction P$.*

Proof. (i)⇒(ii) From [19, 2.2.6]. (iii)⇒(i) Is trivial.

(ii)⇒(iii) There is a triangulation ∇ of the cube C such that the set $\nabla_P = \{T \in \nabla \mid T \subseteq P\}$ is a triangulation of P and is a subdivision of Δ. The existence of ∇ is a well-known fact in polyhedral topology [11,19]. A direct proof can be obtained from an adaptation of the De Concini–Procesi theorem in the version of [17, 5.3]. Actually, by a routine adaptation of the affine counterpart of [9, III, 2.8] we may insist that $\nabla_P = \Delta$. Let $g : C \to [0, 1]$ be the continuous function uniquely defined by the following stipulations: g is linear on every simplex of ∇, g coincides with f at each vertex of ∇_P and $g(v) = 0$ for each vertex v of ∇ not belonging to P. Then $f = g \restriction P$. Evidently, g is piecewise linear. □

For any polyhedron $P \subseteq \mathbb{R}^n$, we denote by $\mathcal{M}_{\mathbb{R}}(P)$ the MV-algebra of all functions $f : P \to [0, 1]$ satisfying any (hence all) of the equivalent conditions (i)–(iii) above.

Now suppose the polyhedron Q is contained in $[0, 1]^n$. As in [15, 4.4] or [17, 9.17], the *semisimple tensor product* $[0, 1] \otimes \mathcal{M}(Q)$ can be identified with the MV-algebra of continuous functions from Q into $[0, 1]$ generated by the *pure tensors* $\rho \cdot g = \rho \otimes g$, where $\rho \in [0, 1]$ and $g \in \mathcal{M}(Q)$.

In Theorem 2.4 we will prove that, up to isomorphism, polyhedral MV-algebras coincide with finitely generated subalgebras of a semisimple tensor product $[0, 1] \otimes \mathcal{M}(R)$, for some *rational* polyhedron $R \subseteq [0, 1]^n$, $n = 1, 2, \dots$. We prepare:

Lemma 2.2. *Up to isomorphism,* $[0, 1] \otimes \mathcal{M}([0, 1]^n) = \mathcal{M}_{\mathbb{R}}([0, 1]^n)$.

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