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A note on axiomatizations of Pavelka-style complete fuzzy logics

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Dedicated to Francesc Esteva, on the occasion of his 70th birthday

Abstract

Pavelka-style completeness, a property relating degrees of provability and truth, was previously studied mainly in the context of logics with continuous connectives. It is known that in some other logics one can use infinitary deduction rule(s) to retain this form of completeness. The present paper offers a systematic study of this idea for fuzzy logics which expand MTL and are given by a fixed standard algebra. We explore the structure of the class of all 'reasonable' expansions of any such logic by rational truth constants and, for several prominent cases, provide axiomatizations of particular expansions enjoying the Pavelka-style completeness.
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1. Introduction

Mathematical Fuzzy Logic (MFL) started as the study of logics based on (left-)continuous t-norms, most prominently Łukasiewicz logic Ł, Gödel–Dummett logic G, Product logic Π, Hájek logic HL,[1] and Esteva–Godo's logic MTL (for a more detailed (historical) account see [1]). These logics are rendered in a language with the truth-constants $\bar{1}$ (truth) and $\bar{0}$ (falsum) and binary connectives $\to$ (implication), $\leftrightarrow$ (equivalence), & (fusion/residuated conjunction), $\wedge$ (lattice conjunction), and $\vee$ (lattice disjunction). They are complete with respect to the *standard semantics*, which has the real-unit interval [0, 1] as the set of truth degrees and interprets the truth constants $\bar{0}$ and $\bar{1}$ by 0 and 1, the pair of connectives &, $\to$ by a left-continuous t-norm and its residuum (which always exists for left-continuous t-norms), and the final pair of connectives $\wedge$, $\vee$ as minimum and maximum. On the other hand, these systems are also complete with respect to an algebraic semantics (MV-algebras, Gödel algebras, product algebras, HL-algebras, and MTL-algebras respectively) and with respect to the subclass of their linearly ordered members, also known as *chains*. In the last years, the scope of MFL has been progressively expanded by changing the basic propositional language by adding new connectives (such as the Monteiro–Baaz $\triangle$ projection, additional negations, conjunctions, or implications; see [8]).

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[1] The logic was introduced by Petr Hájek under the name Basic Logic, BL, and as such was extensively studied by numerous authors (see [3]). However for reasons presented in the recent book chapter [4] we prefer the name *Hájek logic* HL

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In this paper we study expansions of the logics mentioned above by the truth constants from [0, 1] other than $\bar{0}$ and $\bar{1}$. There are two main streams of research in this area of MFL:

- The first one follows the work of Jan Pavelka [21] (actually historically the oldest part of MFL), where one not only adds truth constants for each *real* number from [0, 1] into the language of the Łukasiewicz logic but, more importantly, works with generalized (so-called evaluated) syntax, where the basic syntactical object is a pair ⟨formula, real number⟩. This approach was later elaborated by Vilém Novák and his Ostrava group, see e.g. [17–19], and is thoroughly surveyed in the monograph [20]. Due to the evaluated syntax, even other syntactical and semantical notions, notably those of provability and validity, can be naturally rendered as *graded* (degree-based) concepts and the completeness theorems can be expressed as equality of these two degrees.
- The second approach originated by Petr Hájek in his book [6] where he showed how to interpret Pavelka logic in the framework with traditional syntax. Roughly speaking we can say that he has added truth constants (in his case for rationals from [0, 1] only in order to keep the language countable[2]) but kept the usual syntax. In this setting the 'natural' notion of completeness just identifies the existence of a proof with the validity in all models, thus the equality of *degrees* of provability and validity is an entirely different form of completeness called in Jan Pavelka's honor the 'Pavelka-style completeness'.[3]

It is well known that the first approach is tightly connected to Łukasiewicz logics (as proved already by Pavelka, the continuity of all connectives is crucial for this approach; we will see (indirectly) why it is so in Section 4). On the other hand the second approach can be followed for almost all fuzzy logics and furthermore one need not to restrict to rational truth constants. This research was mainly performed by the Barcelona group under the leadership of Francesc Esteva and has led to numerous papers on the subject, see e.g. [10–12], or an extensive survey of the known results in [8, Section 2].

However most of these papers disregarded the problem of the Pavelka-style completeness, because (as in the first approach) their authors were well aware that it rarely holds for other logics than Łukasiewicz. There are only few exceptions [7,9,14] where the authors propose the use of *infinitary* deduction rules (i.e., the rules with infinitely many premises) to establish the desired Pavelka-style completeness for rational expansions of Product logic, Product-Łukasiewicz logic with Monteiro–Baaz $\triangle$, and the logic RŁΠ. However all these three papers have a serious gap: they take for granted the so-called *Linear Extension Property* (any theory not proving a formula φ can be extended into a linear theory still not proving φ) which is essential for the proof of completeness. The problem is that in the presence of an infinitary rule, one cannot assume that the union of a chain of theories not proving φ still does not prove φ.

The immediate goal of the present paper is to amend the gap: We show (in Lemma 26) that logics with certain infinitary rules still enjoy the Linear Extension Property and thus also the Pavelka-style completeness.[4] The present paper has also one more ambitious goal: To lay foundations of the general study of the Pavelka-style completeness in logics with additional rational truth constants.

We start with a preliminary section which delimits the class of logics whose expansions by rational truth constants we study in this paper; namely the logics $\mathrm{L}_{\boldsymbol{A}}$ given by a particular *standard* algebra $\boldsymbol{A}$ (i.e., algebra with lattice reduct being the real unit interval with the usual order).[5] Then in Section 3 we present, for each logic $\mathrm{L}_{\boldsymbol{A}}$, three prominent (and in general different) such expansions and in Section 4 we study the notion of the Pavelka-style completeness for the class *all possible* rational expansions of $\mathrm{L}_{\boldsymbol{A}}$. The final section provides the axiomatization of the strongest such expansion for some of the logics mentioned above.[6]

[2] It should be noted that even the first approach can be rendered in a countable language, see [17,20].

[3] Petr Hájek has called his logical system *Rational Pavelka Logic*, which is a bit unfortunate terminology, as this logic clearly differs from that of Pavelka in its fundamental nature (the 'genuin' Rational Pavelka Logic was studied in [17]). This terminology was later used by others, referring to expansions of other logics by truth constants. In the present paper we rather speak about 'rational expansions' of a given logic.

[4] The proof of this lemma is based on the proof of Lemma 3.4.25 from my PhD thesis [2] and also the whole paper can be seen as elaboration of Subsection 3.4 of [2], with the main characterization Theorem 22 being generalization of [2, Theorem 3.4.21] and main axiomatization Theorems 27 and 28 being generalizations of [2, Theorem 3.4.23 and Corollary 3.4.26].

[5] It should be noted that there are works (e.g. [13,16]) studying the Pavelka-style completeness for logics based on algebras which are not standard.

[6] As we show in Section 4, the maximal rational expansion (of any $\mathrm{L}_{\boldsymbol{A}}$) always enjoys the Pavelka-style completeness and our axiomatizations coincide with those used in the literature, our result can be read as the promised proof of the Pavelka-style completeness for those logics.

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