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Non-commutative first-order EQ-logics ☆,☆☆

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Abstract

EQ-algebras are meet semilattices endowed with two additional binary operations: *fuzzy equality* and *multiplication*. These algebras serve as a basic algebraic structure of truth values for many-valued logics based on (fuzzy) equality instead of implication. Therefore, they generalize residuated lattices in the sense that each residuated lattice is an EQ-algebra but not vice-versa.

Logics based on EQ-algebras are called EQ-logics and they can be considered as special kind of fuzzy logics. After developing propositional and higher-order ones, we address in this paper the predicate first-order EQ-logic. First, we overview some basic properties of EQ-algebras and the basic propositional EQ-logic.

Analysis of necessary properties of the fuzzy equality that is in predicate EQ-logic considered not only between truth values (the equivalence) but also between objects revealed that we cannot consider the fuzzy equality in full generality without means enabling us to deal with the classical (crisp) equality. This is possible using the $\boldsymbol{\Delta}$-connective. Therefore, we pay a special attention to prelinear EQ$_\Delta$-algebras and develop the corresponding propositional EQ$_\Delta$-logic. Finally, we in detail introduce syntax and semantics of the first-order EQ-logics and prove various theorems characterizing its properties including completeness.

Keywords: EQ-algebra; EQ-logic; Equational logic; Delta connective

1. Introduction

In [17], the concept of EQ-algebra was introduced. It was motivated by the study of higher-order fuzzy logic [16]. Recall that the latter generalizes the simple type theory in the style of L. Henkin who developed in [14] a very elegant theory (cf. also [1]) in which the basic connective is equality. His theory can be taken also as a possible answer to G.W. Leibniz who required that “a fully satisfactory logical calculus must be an equational one” (cf. [2]).

☆ This paper is dedicated to Prof. Fracesc Esteva at the occasion of his 70th birthday. Let us remind that the concept of EQ-logic was first introduced in the conference held in Barcelona at the occasion of Francesc’s 65th birthday. The predicate EQ-logic thus naturally crowns the research inspired by him.

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The paper [16] follows the Henkin's idea. The role of the algebra of truth values was taken there by a residuated lattice (i.e., an integral, bounded, commutative, residuated, lattice ordered monoid). The conceptual problem raised with this algebra, however, is the fact that the basic operation in it is residuation that in fuzzy logic serves as a natural interpretation of implication. Then interpretation of equivalence is a derived operation of biresiduation: $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. But this contradicts the Leibniz's requirement that the equality (equivalence) should be basic while implication should be derived. A natural question is raised whether there exists also an algebra for many-valued (fuzzy) logic in which fuzzy equality is the basic operation. A possible answer is the concept of EQ-algebra, on the basis of which a special fuzzy type theory was developed in [18].

One of the questions that had to be answered from the beginning was, whether EQ-algebras bring anything new in comparison with the residuated lattices? In other words, isn't it only a residuated lattice in disguise? The answer is *no*. EQ-algebras are indeed more fundamental than the residuated lattices in the sense that every residuated lattice is an EQ-algebra but not vice-versa. A more detailed study of them (cf. [7,8,19]) reveals intrinsic interrelations among various kinds of properties of this algebra and also the role of residuation in them. Recall that in EQ-algebra, only a semi-residuation holds: if $a \leq b \to c$ then $a \otimes b \leq c$. Therefore, we can ask, when the EQ-algebra becomes fully residuated, i.e., when $\langle \otimes, \to \rangle$ becomes a residuated couple of operations. The answer is given in [8, Proposition 6]. It is interesting that among conditions equivalent with residuation is the following:

$$a \to b \leq a \otimes c \to b \otimes c \tag{1}$$

for all a, b, c. At the same time, a similar condition

$$a \sim b \leq a \otimes c \sim b \otimes c \tag{2}$$

implies residuation but is not equivalent with it, only (2) implies (1) but not vice-versa. It is easy to see that if the algebra is linearly ordered then both inequalities (1) and (2) are equivalent to each other.

Having answered the question whether EQ-algebra brings anything new, we meet another essential question: what conditions must be fulfilled by an EQ-algebra to make it possible to develop many-valued logic with complete syntax. The resulting logic will be called EQ-logic and it can be ranked among special kinds of fuzzy logics.

Because the motivation for establishing EQ-algebra was to develop higher-order fuzzy logics based on such algebra of truth values, it was natural that higher-order EQ-logic was developed as the first step. We found that the principal properties that clearly are fulfilled by all the residuated lattices but are not there formulate as explicitly as in EQ-algebras are *separateness*, i.e., $a \sim b = 1$ iff $a = b$, and *goodness*, i.e., $a \sim \mathbf{1} = a$. The goodness property implies separateness but not vice-versa. Separateness turned out to be indispensable not only for higher-order logic but for any kind of logic. This means that it is not possible to relax equivalence in such a way that different truth values can be equivalent (equal) in the degree 1. On the other hand, the goodness property is equivalent with validity of the inequality $a \otimes (a \to b) \leq b$ that is the necessary condition for soundness of modus ponens inference rule in fuzzy logic.

The EQ-algebra of truth values for higher-order fuzzy logic must necessarily be linearly ordered (note, however, that linearity is not enough strong to enforce residuation). This suggests a question whether we can find weaker EQ-algebras for logics with complete syntax. The answer is positive but only for propositional EQ-logics and was published in [6]. We conclude that a reasonable logical calculus requires the fuzzy equality to behave in limit cases as classical equality and so, generalization of the equality cannot go too far. Still, we developed EQ-logics that are more fundamental than MTL-logic[1] because few more axioms must be added to basic EQ-logic to obtain logic equivalent with MTL (cf. [6, Theorem 9]).

The necessity to preserve classical equality within fuzzy one manifests itself even more strongly. First of all, it turned out that we cannot prove deduction theorem without introducing the $\boldsymbol{\Delta}$-connective. But the latter enables us to pick up boolean substructure from the given EQ-algebra (residuated lattice) and so, the formula $\boldsymbol{\Delta}(A \equiv B)$ is just classical equivalence (i.e., classical equality between truth values).

To complete the picture of EQ-logics, it is necessary to develop also its first-order version. Apparently, the $\boldsymbol{\Delta}$-connective is necessary there, too and so, we conclude that when starting with fuzzy equality (equivalence) as the basic connective, we must also consider the $\boldsymbol{\Delta}$ connective in all levels of EQ-logics.

[1] The basic fuzzy logic considered so far.