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Co-rotation-annihilations of residuated semigroups ☆

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Abstract

Two new ways to construct involutive residuated semigroups are introduced in this paper, namely, connected and disconnected co-rotation-annihilations. Co-rotation-annihilations utilizes two particular kinds of residuated semigroups to construct a third one. The method is suitable for constructing a large class of examples of negative rank involutive FL_e-algebras, that is, where the unit element of the algebra is smaller than the falsum constant, which defines the involution in the algebra.

Keywords: Involutive residuated semigroup; Constructions; Substructural logic

1. Introduction

A classification of absorbent-continuous group-like FL_e-algebras over subreal chains has been presented in [9]. It has been generalized to all order-dense chains in [3]. At this point, several research problems seem plausible: to describe group-like FL_e-algebras over densely ordered chains (that is, to drop the absorbent-continuity condition) or to describe absorbent-continuous group-like FL_e-algebras over arbitrary chains. Also, since group-like algebras are just zero rank algebras, a structural description of positive and/or negative rank algebras seems to be a plausible next step. The rotation constructions and the rotation-annihilation constructions can be used to construct a huge class of positive rank algebras. However, hardly any examples have been known for negative rank ones. The introduction of the co-rotation constructions has made a significant leap in this direction [4].

The connected rotation construction for t-norms has been introduced in [7] (see [4] and the references therein), and has been generalized to arbitrary posets in [5], leading to the connected and the disconnected rotation constructions. These constructions have proved crucial in the structural description of, for example, perfect and bipartite IMTL-algebras [13], free nilpotent minimum algebras [1], and free Glivenko algebras [2]. Recently, two other constructions, called disconnected and connected *co*-rotation, have been introduced in [4] along with the characterization of the semigroups that can be utilized in them. The connected rotation-annihilation construction for t-norms has

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been introduced in [8] and has been generalized to arbitrary posets in [5] leading to the disconnected and the connected rotation-annihilation constructions. The aim of this paper is to make the picture complete by introducing and investigating connected and the disconnected *co*-rotation-annihilation constructions. Co-rotation-annihilation is a construction which, from two residuated semigroups of particular types constructs a third residuated semigroup, which is involutive, and acts on an ordinal sum of posets both involving and derived from the original partially ordered universes. Over complete and order-dense chains, these constructions can be considered 'skew dual' to the respective rotation-annihilation constructions, cf. [6, Section 3]. Just as the rotation-annihilation constructions, when using FL_e-algebras as their first starting operations, result in positive rank involutive FL_e-algebras, the co-rotation-annihilation constructions, when using FL_e-algebras as their first starting operations, result in negative rank involutive FL_e-algebras; thus providing a wide spectrum of examples for the latter algebra.

Rotation, co-rotation, rotation-annihilation and co-rotation-annihilation constructions can also be viewed as some kind of geometric procedures, which put together a new associative operation using some starting operations, see for example the figures of this paper. Similar geometric-flavored constructions have a considerable history in this field [7,8,10–12,14,15].

A po-monoid (or its monoidal operation) is called *integral* if it has a top element which is also the unit element of the semigroup operation. A poset is involutive if there exists and involution over its universe, that is, $' : X \to X$ such that $x'' = x$. A po-semigroup *with an involution* is a po-semigroup over an involutive poset such that the involution is order-reserving, that is, $x \le y$ implies $x' \ge y'$. A po-semigroup with an involution $(X, \circledast, ')$ is called *rotation-invariant* (with respect to its involution $'$), if, for all $x, y, z \in X$, $x \circledast y \le z'$ if and only if $y \circledast z \le x'$. A commutative binary operation $\circledast$ on a poset $(X, \le)$ is called *residuated* if there exists another binary operation $\to_\circledast$ on X such that, for $x, y, z \in X$, $x \circledast y \le z$ iff $y \to_\circledast z \ge x$. A residuated poset with least element is called *conjunctive* if, for $x, y \in X$, $x \circledast y \le x$ and $x \circledast y \le y$. A residuated poset with least element is called *weakly disjunctive* if, for $\bot \ne x, y \in X$, $x \circledast y \ge x$ and $x \circledast y \ge y$. $\mathcal{U} = \langle X, \le, \circledast, \to_\circledast, t, f \rangle$ is called an *FL_e-monoid* if $\mathcal{C} = \langle X, \le \rangle$ is a poset, $(X, \circledast, \to_\circledast, t)$ is a commutative, residuated monoid over $\mathcal{C}$, and f is an additional constant. If $(X, \le)$ is a lattice, we speak about *FL_e-algebras*.[1] If X is linearly ordered, we speak about FL_e-*chains*. *Commutative residuated lattices* are exactly the f-free reducts of FL_e-algebras. Call $\mathcal{U}$ *involutive*, if

$$\text{for } x \in X,\ (x')' = x \text{ holds, where } x' = x \to_\circledast f. \tag{1}$$

We say that $\mathcal{U}$ has positive (reps. non-positive) rank if $t > f$ (resp. $t \le f$). Note that the class of bounded involutive FL_e-monoids with $f = \bot$ coincides with the class of Girard monoids [5]. The other extreme situation is when $t = f$, in which case we call the involutive FL_e-monoid (and also its monoidal operation) *group-like*. The class of absorbent-continuous group-like FL_e-chains has been classified in [9]. *Uninorms* are monoidal operations of FL_e-monoids on [0, 1]. Integral uninorms are called *t-norms*.

Lemma 1. *(See [5].) Any rotation invariant operation is residuated.*

Finally, the two rotation-annihilation constructions are recalled here from [5] to provide the reader with a solid comparison with their co-rotation-annihilation counterparts to be introduced in this paper. Denote the class of posets with greatest element, posets with least element, the class of bounded posets and the class of involutive posets, the class of bounded involutive posets by $\mathcal{M}^\top$, $\mathcal{M}^\bot$, $\mathcal{M}^{\top\bot}$, $\mathcal{M}^{inv}$ and $\mathcal{M}^{inv\top\bot}$, respectively.

1.1. Rotation-annihilation constructions

Definition 1.

A. (ω-**operator**) We define the operator $\omega : \mathcal{M}^\top \times \mathcal{M}^{inv\top\bot} \to \mathcal{M}^{inv\top\bot}$ the following way. Let $M_1 \cap M_2 = \emptyset$, $(M_1, \le, \top)$ be a poset with greatest element and $(M_2, \preceq, ^*)$ be an involutive poset. Take a copy $M_1{}^{‘}$ of M_1 which is disjoint from M_1 and M_2 and equip it with the dual relation of $\le$. That is, $M_1{}^{‘} := \{x^{‘} \mid x \in M_1\}$ and for $x^{‘}, y^{‘} \in M_1{}^{‘}$, $x^{‘} \le^{‘} y^{‘}$ if and only if $y \le x$. Let $\omega(M_1, M_2) = M_1 \cup M_2 \cup M_1{}^{‘}$ and define a partial ordering $\le_\omega$ on $\omega(M_1, M_2)$ as follows. For $x, y \in \omega(M_1, M_2)$, let $x \le_\omega y$

[1] In the definition of FL_e-algebras one replaces the partial order relation in the signature by the lattice operations.