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The mean square error of a random fuzzy vector based on the support function and the Steiner point ☆

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Abstract

Metrics between fuzzy values are a topic with interest for different purposes. Among them, statistics with fuzzy data is growing in modelling and techniques largely through the use of suitable distances between such data. This paper introduces a generalized (actually, parameterized) L^2 metric between fuzzy vectors which is based on their representation in terms of their support function and Steiner points. Consequently, the metric takes into account the deviation in 'central location' (represented by the Steiner points) and the deviation in 'shape' (represented by a deviation defined in terms of the support function and Steiner points). Then, sufficient conditions can be given for this representation to characterize fuzzy vectors, which is valuable for different aims, like optimization studies. Properties of the metric are analyzed and its application to quantify the mean (square) error of a fuzzy value in estimating the value of a fuzzy vector-valued random element is examined. Some immediate implications from this mean error are finally described.

Keywords: Fuzzy vector; Mean square error; Random fuzzy vector; Steiner point; Support function

1. Introduction

Distances between fuzzy sets are a topic which has received a deep attention in the literature. On the one hand, it has been considered in connection with studies of similarity between fuzzy sets (see, for instance, Dubois et al. [14], Beg and Ashraf [2], and Esteva et al. [15]). On the other hand, for statistical purposes such as classification of fuzzy-valued

☆ This paper has been prepared in honor to our friend Françesc Esteva on the occasion of his (SOMEWHAT ELDERLY) birthday. Although we have never worked directly on his research field, his contributions to fuzzy logic have certainly benefited our own.

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elements or inferential statistics with fuzzy set-valued random elements, distances play a key role (see, for instance, Blanco-Fernández et al. [4,5]).

In several papers (e.g., Bertoluzza et al. [3], Casals et al. [7], Trutschnig et al. [33]) the suitability of considering metrics between fuzzy vector values taking into account both their 'central location' along with their 'shape' has been pointed out and discussed.

For some practical goals, often related to optimization tasks, it can also be convenient to use metrics based on representations of fuzzy vector values for which there exist sufficient conditions characterizing fuzzy vectors.

A well-known distance satisfying the last desirable property is the metric ρ_2 (see Diamond and Kloeden [10]), which is based on the support function of the involved fuzzy vectors and takes into account the usual inner product for Hilbert space functional values. When fuzzy vectors reduce to fuzzy numbers, this metric corresponds to the one based on the infimum/supremum characterization of fuzzy numbers.

Recently (see Sinova et al. [28]), a new parameterized representation of fuzzy numbers, along with an associated L^2 metric, has been introduced. This representation describes each fuzzy number by means of an indicator of its central location and an indicator (in fact, two) of its shape, and there exist sufficient conditions characterizing fuzzy numbers. This is the essential difference with Bertoluzza et al.'s metric [3] (see also Trutschnig et al. [33]): there don't exist sufficient conditions for the mid-point/spread representation of fuzzy numbers behind such a metric characterizing fuzzy numbers.

This paper aims to extend the same idea behind this metric to the fuzzy vectorial-valued case. The rest of the paper is organized as follows: Section 2 recalls some basic concepts on fuzzy vectors and the suggested representation in terms of the support function and the generalized Steiner points; Section 3 introduces an L^2 metric based on the representation in Section 2 and its properties are examined; Section 4 states a measure of the mean square error based on the new metric, and some applications are indicated in Section 5.

2. The representation of fuzzy vectors through their support function and generalized Steiner point

Let $\mathcal{F}_c^*(\mathbb{R}^p)$ denote the space of (bounded) fuzzy vectors of $\mathbb{R}^p$, where a (bounded) *fuzzy vector* $\widetilde{U} \in \mathcal{F}_c^*(\mathbb{R}^p)$ is a fuzzy set of $\mathbb{R}^p$, that is, a mapping $\widetilde{U} : \mathbb{R}^p \to [0, 1]$ such that it is normal and fuzzy-convex, upper semi-continuous and its support set is bounded. Equivalently, for all $\alpha \in [0, 1]$ the *α-level* of $\widetilde{U}$, defined as

$$\widetilde{U}_\alpha = \begin{cases} \{\mathbf{x} \in \mathbb{R}^p : \widetilde{U}(\mathbf{x}) \geq \alpha\} & \text{if } \alpha \in (0,1] \\ \text{cl}\{\mathbf{x} \in \mathbb{R}^p : \widetilde{U}(\mathbf{x}) > 0\} & \text{if } \alpha = 0, \end{cases}$$

is a nonempty compact subset of $\mathbb{R}^p$.

On the space $\mathcal{F}_c^*(\mathbb{R}^p)$ one can consider the usual *fuzzy arithmetic* based on Zadeh's extension principle [37]. Given $\widetilde{U}, \widetilde{V} \in \mathcal{F}_c^*(\mathbb{R}^p)$ and $\gamma \in \mathbb{R}$, the *sum of* $\widetilde{U}$ *and* $\widetilde{V}$ is defined as the fuzzy vector $\widetilde{U} + \widetilde{V} \in \mathcal{F}_c^*(\mathbb{R}^p)$ such that

$$(\widetilde{U} + \widetilde{V})(\mathbf{t}) = \sup_{\mathbf{y}+\mathbf{z}=\mathbf{t}} \min\{\widetilde{U}(\mathbf{y}), \widetilde{V}(\mathbf{z})\}$$

or, equivalently and based on Nguyen [23], for each $\alpha \in [0, 1]$:

$$(\widetilde{U} + \widetilde{V})_\alpha = \text{Minkowski sum of } \widetilde{U}_\alpha \text{ and } \widetilde{V}_\alpha = \{\mathbf{y} + \mathbf{z} : \mathbf{y} \in \widetilde{U}_\alpha, \mathbf{z} \in \widetilde{V}_\alpha\}.$$

The *product of* $\widetilde{U}$ *by the scalar* γ is defined as the fuzzy vector $\gamma \cdot \widetilde{U} \in \mathcal{F}_c^*(\mathbb{R}^p)$ such that

$$(\gamma \cdot \widetilde{U})(\mathbf{t}) = \sup_{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} = \gamma \mathbf{t}} \widetilde{U}(\mathbf{y}) = \begin{cases} \widetilde{U}(\mathbf{t}/\gamma) & \text{if } \gamma \neq 0 \\ \mathbf{1}_{\{0\}}(\mathbf{t}) & \text{if } \gamma = 0 \end{cases}$$

or, equivalently and based on Nguyen [23], for each $\alpha \in [0, 1]$:

$$(\gamma \cdot \widetilde{U})_\alpha = \gamma \cdot \widetilde{U}_\alpha = \{\gamma \cdot \mathbf{y} : \mathbf{y} \in \widetilde{U}_\alpha\}.$$

It is well-known that when $\mathcal{F}_c^*(\mathbb{R}^p)$ is endowed with the two preceding operations we get a semi-linear but not a linear space, since $\widetilde{U} + (-1) \cdot \widetilde{U} \neq \mathbb{1}_{\{\mathbf{0}\}}$ (neutral element for the fuzzy sum), but in case $\widetilde{U}$ reduces to the indicator function of an element in $\mathbb{R}^p$. As a consequence of this, one cannot state an extension of the difference on the space $\mathcal{F}_c^*(\mathbb{R}^p)$ which can simultaneously be always well-defined and preserve the properties of the difference with real numbers in connection with the sum. This concern motivates and strongly reinforces the interest in having suitable metrics between fuzzy vectors, especially in developing certain statistics with them.