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A naïve way of looking at fuzzy sets

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To Francesc, with our friendship, and with our esteem for the work of professor Esteva.

Abstract

In this study, we consider the concept of a predicate (P) in a universe of discourse X from a specific viewpoint, i.e., the informational viewpoint with respect to its linguistic use. Its meaning and its different types are considered, particularly by considering the predicates that are "measurable" and designate a "collective" $(\boldsymbol{P})$ in X, which is not always a classical subset of X. We show that the collective $\boldsymbol{P}$ manifests itself in different "states" or fuzzy sets, where knowledge and representation depend on the available information regarding the use of the predicate P in X. We also analyze the linguistic concept of a "collective" where the fuzzy sets are nothing other than informational states, something that is interesting from the viewpoint of their design, at least conceptually. We also indicate the relevance of other points and questions that have loose but basic relationships with the main problem addressed.

Keywords: Basic concepts; Linguistic modeling; Mathematics; Fuzzy sets

"The form of the statements in both Language Games
is the same: X is <u>more clear than</u> Y."
Ludwig Wittgenstein, *Remarks on Colour*

1. Introduction

People with a specific mathematical background tend to view fuzzy sets as a generalization of crisp sets, where the algebraic structure is that of a Boolean algebra, although it is not the case that any fuzzy set always has a crisp part. It could be said that this tendency is natural because science, and particularly mathematics, is based on Boolean calculus using such sets, which leads to the belief that anything may be specified using them. This is a belief that comes mainly from the frequent but not explicitly stated use of the so-called Specification Axiom [1], according to

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which any binary predicate acting upon a crisp set is specified as a crisp subset of the universe and its corresponding complement.

However, the Boolean algebra structure of subsets has many rules, which include valid properties that are not universally valid in natural language [2]. This is why the Specification Axiom is not always valid with predicates in the same manner as in language. Ordinary, everyday, or natural language is charged with many linguistic terms, which, as shown by the Sorites methodology [3,4] when acting in a crisp set X, lack the specification of a crisp subset, but they may be specified by means of fuzzy sets [5]. Indeed, this study aims to answer the question: "*what does it mean to "specify" a non-crisp predicate?*"

This is a question related to predicates P such that the meaning of the elementary statements "*x is P*," which describe their behavior in X (where X is a crisp universe of discourse and x is in X), does not admit a definition such as "*if and only if*" (*iff*), for instance, as in the case of the binary predicate $P = algebraic$ in the set X of real numbers, which specifies the set of algebraic real numbers. (Recall that a real number x is algebraic if and only if it is a root of a non-zero polynomial in one variable with integer coefficients.) Note that a predicate P is simply a name given to a property p exhibited by the elements of X and many properties that are usual in a natural language only appear in a graded form. Elementary statements with non-binary or gradable predicates cannot be described as binaries, but in the best case this can be achieved using a finite family of rules (conditional statements), which are insufficient to determine a crisp specification of the predicate in the corresponding universe of discourse. Thus, they can only indicate that if "*x is P*," then such rules must be valid, i.e., the rules are necessary but not sufficient. It should be noted that although the elements of X may have a real or virtual character, the elementary statements are simply linguistic expressions with no more reality than that which they pretend to mean.

Example 1. Let $X = [0, 1000]$ and consider the predicate $M =$ *greater than or equal to 770*. Then, it holds that:

$$\text{"}x \text{ is } M\text{" if and only if } x \geq 770, \text{ if and only if } x \in [770, 1000].$$

This specification is the interval $\boldsymbol{P} = [770, 1000]$ within X, which is perfectly classified by

$$X = [0, 770) \cup [770, 1000],$$

(where $[0, 770) = [770, 1000]^c$), and it expresses that a number x of X can either be greater than or equal to 770 or not. This means that the use of M is binary. No other elementary statement of "*x is M*" can be stated other than those with x in $\boldsymbol{P}$, thus those with x in $\boldsymbol{P}^c$ cannot be stated.

However, if a typical natural language predicate $G = big$ in the same numerical universe $[0, 1000]$ is considered, the situation is radically different from the former, because it is not possible to define the meaning of "*x is G*" in an equivalent form or with other precise statements, i.e., in an "*iff*" form. Therefore, it does not necessarily exist a classical subset $\boldsymbol{G}$ of $[0, 1000]$, such that assuring that "*x is G*" is equivalent to state "$x \in \boldsymbol{G}$." However, since G is used in an ordinary language, it is convenient to explain how "*x is G*" can be stated in X. First, it appears to be natural to assure that "*1000 is G*," but not that "*0 is G*." Second, if it can be stated that "*x is G*" and there exists some y in X that is equal to or *bigger* than x ($x \leq y$), then it can also be stated that "*y is G*."

These two rules indicate how G is basically used in $[0, 1000]$ and, without any doubt, they have to be applied correctly to express the meaning of G in X. In other words, any way of using *big* in X should at least be in agreement with the rules. This does not exclude the possibility that some uses of G may verify some additional rules. This may be the case, for instance, when the numbers from 0 to 1000 refer to data in some specific situation, e.g., hectares in a field in an agriculture context and light years in an astronomical context.

Obviously, the former predicate M represents a particular constrained use of G since it verifies the two rules mentioned earlier. However, if a new version $G^* = big$ refers, for example, to the weights of chemical substances between 0 and 1000 milligrams, although X would still be the same interval $[0, 1000]$, its elements would indicate weights in milligrams instead of simple numbers and it may be appropriate to add a third rule, as follows.

If it can be assured that "*x is G**" (where $0 < x < 1000$), then the numbers $10^{-n(x)}$ and $10^{-m(x)}$ exist such that it can be assured that "$x + 10^{-n(x)}$ *is G**" and also that "$x - 10^{-m(x)}$ *is G**," where the values of $n(x)$ and $m(x)$ are natural numbers, which may be independent of the point x. In a particular case, one or both functions may be constant, or they may take the same value for some x. However, M does not represent a particular use of G^*, since if it were possible to assure that "*770 is G**," it would also be possible to assure that