Available online at www.sciencedirect.com

SciVerse ScienceDirect

Fuzzy Sets and Systems 239 (2014) 91–103

FUZZY sets and systems

www.elsevier.com/locate/fss

A method for deriving order compatible fuzzy relations from convex fuzzy partitions

Sandra Sandri*, Flávia Toledo Martins-Bedê

Brazilian National Institute for Space Research (INPE), Av. dos Astronautas, 1758, S.J. Campos, S.P., Brazil

Available online 7 December 2013

Abstract

We address a special kind of fuzzy relations capable of modeling that two elements in the universe of discourse are similar to the extent that they are close to each other with respect to a given total order. These order compatible fuzzy relations are reflexive and symmetric but not necessarily T-transitive. We address the requirements to construct such relations from a large class of fuzzy partitions that obey some requirements from the fuzzy sets in the partition, such as convexity, that are useful but not severely constraining. We also propose a new method to obtain those fuzzy relations from these partitions.

Keywords: Similarity relations; Fuzzy relations; Fuzzy partitions; Total order; T-indistinguishable operators

1. Introduction

Several means can be devised to create fuzzy relations that somehow model the concept of similarity for use in applications. The user can enter the relation directly using a look-up table, when the application domain is discrete. In problems involving continuous domains, the relation can be obtained by choosing a family of parametrized relations, whose parameters are learnt from data or directly specified. A more user-friendly possibility consists in deriving the fuzzy relation out of a finite collection of fuzzy sets. The fuzzy sets themselves can be easily elicited from the user or learnt directly from the data. The present work addresses the problem of obtaining relations from fuzzy sets for applications in which the notion of similarity between two elements of a domain is based on their closeness on that domain.

In classical set theory, the notion of similarity between elements of a given domain can be modeled using two formalisms: a partition (a collection of non-intersecting sets covering the domain) and an equivalence relation (a reflexive, symmetric and transitive relation). The two formalisms are equivalent: from a given partition, one determines a unique equivalence relation and vice versa.

In the fuzzy framework, a fuzzy partition $\mathbf{A}$ on a given domain Ω is usually referred to as a collection of fuzzy sets of Ω at least covering (in some way) the domain. Several specific kinds of fuzzy partitions exist in the literature (see e.g. [20]) and they are all capable of capturing that two elements with positive membership degrees to a given fuzzy subset $A_i \in \mathbf{A}$ are somewhat related, thus modeling a weak notion of similarity.

In what regards modeling similarity by means of fuzzy relations, the most well-established approach is the one that requires a fuzzy relation to be reflexive, symmetric and to obey T-transitivity, the fuzzy counterpart of transitivity.

* Corresponding author. Tel.: +55 12 32086852.
E-mail addresses: sandra.sandri@inpe.br (S. Sandri), flavinha@lac.inpe.br (F.T. Martins-Bedê).

0165-0114/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.fss.2013.11.011

Indeed, T-transitivity plays the same role of transitivity in crisp relations; relations obeying it are thus adequate to model similarity in the fuzzy framework as equivalence relations do in the crisp counterpart.

T-transitivity is however ill-suited to deal with a notion of similarity in which two elements are similar to the extent that they are close to each other in the domain in which they are defined. For example, let us consider a relation that models the concept of "sweetness" [16]. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ stand for the number of spoons of sugar to be poured into a cup of coffee, with $\omega_1 < \omega_2 < \omega_3$. As Ω is ordered, it is natural to expect that the similarity between the elements of the pair (ω_1, ω_3) is smaller than that existing between either (ω_1, ω_2) or (ω_2, ω_3) (see [16,22]). In this case, transitivity does not play a role, as the issue is more related to the notion of a total order than of an equivalence relation. Tversky [28] brings a deep discussion on how similarity is perceived by human beings and questions the use not only of transitivity, but also of reflexivity and symmetry. In the specific context of fuzzy relations, several authors addressed the inadequacy of T-transitivity (see e.g. [5,12,22]).

In previous works [7,11,17,10], we have dealt with a real-world problem using fuzzy relations for which transitivity does not play a role. The relations used in these works are reflexive, symmetric and obey an extra property: two numbers are similar to the extent that they are close to each other with respect to the Euclidean distance [9] (see also [18,23]). Obtaining these relations is rather difficult, which contrasts with the ease with which fuzzy partitions can be elicited. This paper aims at obtaining this type of relation from collections of fuzzy sets and is an extension to a preliminary work on the subject, presented in [21].

The transformations between a collection of fuzzy sets and T-transitive relations, defined on the same domain, have been established by *Representation Theorem* [29] (see also [24,25,4]). This problem has also been addressed when transitivity is modeled using other operators than T-norms (see e.g. [13]). However, the transformation between fuzzy partitions and fuzzy relations in which similarity is based in closeness on an ordered domain, as described above, is yet to be addressed.

The literature addressing the modeling of similarity in the fuzzy framework is very rich. Lotfi Zadeh defined a binary fuzzy relation S as a *similarity relation* on Ω when S is reflexive, symmetric and min-transitive [30]. This term was later generalized by replacing min-transitivity by T-transitivity (see e.g. [24]). Relations that are symmetric, reflexive and T-transitive have also been called *T-indistinguishable operators* (see [19]) or *T-equivalence relations* (see [13]). Moreover, general terms have been used to name some of these relations, depending on the T-norm used to create them (see [19]). Relations derived from Lukasiewicz T-norm have been named *likeness relations*, whereas those using the product have been called *possibility relations* [19,29] and *probabilistic fuzzy relations* [15]. Relations that are reflexive and symmetric but not necessarily T-transitive are known as *proximity* or *tolerance* relations (see e.g. [19]). Some authors call *tolerance relations* the symmetric fuzzy relations that obey a weak reflexivity property (see e.g. [3]). Other types of fuzzy relations include *closeness relations* [8] and *coherent nearness relations* [6].

As we see, the terminology in this domain is somewhat confusing, because generic terms from the English language vocabulary are used to name fuzzy relations with very specific properties. The profusion of closely related generic terms makes it hard to obtain an accurate translation of all of them in some languages. We prefer to adopt more specific/technical (however clumsy) terms in order to name concepts that rely on choices of properties, rather than general ones, yet keeping the semantic flavor.

We use the specific terms *T-indistinguishable relations* for those relations that are reflexive, symmetric and T-transitive [19] and the term *Order Compatible Fuzzy Relations* (OCFR) for those that are reflexive, symmetric and compatible with respect to a total order. We also present our own definition of fuzzy partitions, called *Convex Fuzzy Partitions with Respect to a Total Order* (CFP), whose fuzzy sets are convex with respect to a given total order.

This paper is organized as follows. In the following section we give well-established definitions that are used throughout the text. In Sections 3 and 4 we introduce Convex Fuzzy Partitions with Respect to a Total Order and Order Compatible Fuzzy Relations, respectively. In Section 5, we study some properties on transformation methods to generate OCFRs from CFPs. In Section 6, we propose one method to generate an OCFR from a CFP and discuss the particular case of Ruspini partitions. Section 7 finally brings the conclusions with some guidelines for future work.

2. Basic definitions

A fuzzy set A on a domain Ω is a mapping $A : \Omega \to [0, 1]$. It is said to be normalized when $\exists x_0 \in \Omega$ such that $A(x_0) = 1$. The core and support of a fuzzy set A are defined as $core(A) = \{x \in \Omega | A(x) = 1\}$ and $supp(A) = \{x \in$