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ScienceDirect
Fuzzy Sets and Systems 280 (2015) 27–57
FUZZY sets and systems
www.elsevier.com/locate/fss

On the Ulam stability of first order linear fuzzy differential equations under generalized differentiability

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Received 28 July 2014; received in revised form 7 January 2015; accepted 8 January 2015

Available online 13 January 2015

Abstract

In this paper we investigate, under some suitable conditions and generalized differentiability, the Ulam stability problems of three variants of first order linear fuzzy differential equations, respectively.

Keywords: Ulam stability; H-difference; Generalized differentiability; Linear fuzzy differential equations

1. Introduction

Nowadays, the Ulam stability is gradually becoming one of the most active research topics in the theory of functional equations. The study of such stability problems of functional equations originated from a question of Ulam [35] concerning the stability of the group homomorphisms, i.e.,

Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ such that $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the following year, Hyers [12] gave a first affirmative partial answer to this question for Banach spaces. Later, the theorem of Hyers was generalized by T. Aoki [3] for additive mappings and by Rassias [31] for linear mappings by considering an unbounded Cauchy difference. Since then, the stability problems for different types of functional equations in various abstract spaces have been extensively studied [16]. It is worth noting the pioneering work of Mirmostafaee et al. [20,21], which studies Cauchy and Jensen functional equations in fuzzy normed spaces. It should

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http://dx.doi.org/10.1016/j.fss.2015.01.002

be mentioned that our research of Ulam stability from a different perspective of fuzziness was inspired by the idea presented in [20,21].

In 1990s, Obloza [28] has initiated the study of the Ulam stability of differential equations. Afterwards, Alsina and Ger [2] have proved the Hyers–Ulam stability of the differential equation $y' = y$. More precisely, for a given $\epsilon > 0$, if f is a differentiable function from an open interval I into $\mathbb{R}$ with $|f'(t) - f(t)| \leq \epsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ such that $g'(t) = g(t)$ which satisfies $|f(t) - g(t)| \leq 3\epsilon$ for all $t \in I$. The Ulam stability of the differential equation $y' = \lambda y$ in various abstract spaces has been further considered by Miura and Takahasi et al. [22,23,33]. So far, the Ulam stability problems of the first order and higher order linear differential equations have been widely and extensively studied by various authors [1,8,13–15,24,29,32,34].

Generally speaking, the Ulam stability includes the Hyers–Ulam stability and Hyers–Ulam–Rassias stability. Let X be a Banach space with the norm $\|\cdot\|$. For an n-order X-valued differential equation

$$F\big(t, y, y', \ldots, y^{(n)}\big) = 0, \quad t \in I_1,$$

where I_1 denotes a subinterval of $\mathbb{R}$, we say that it has Hyers–Ulam stability or it is stable in the sense of Hyers–Ulam if for a given $\epsilon > 0$ and an n times strongly differentiable mapping $f : I_1 \to X$ satisfying $\|F(t, f, f', \ldots, f^{(n)})\| \leq \epsilon$ for all $t \in I_1$, there exists an exact solution $g : I_1 \to X$ of the preceding differential equation such that $\|f(t) - g(t)\| \leq K(\epsilon)$ for all $t \in I_1$, where $K(\epsilon)$ depends only on ϵ and $\lim_{\epsilon \to 0} K(\epsilon) = 0$. More generally, if the ϵ and $K(\epsilon)$ are replaced by two control functions φ and Φ in t, respectively, then we say that the differential equation mentioned above has the Hyers–Ulam–Rassias stability or it is stable in the sense of Hyers–Ulam–Rassias.

Unlike the general stability of differential equations, the Ulam stability can guarantee the existence (and even uniqueness) of the exact solution of a differential equation, provided that an approximate solution with a determined error is given. Conversely, it is not difficult to see that the solution is stable for a differential equation with the Ulam stability. Therefore, the Ulam stability not only establishes an important foundation for the existence (and even uniqueness) of the solution of differential equations, but also provides a reliable theoretical basis for approximately solving differential equations.

As a natural way to model dynamical systems with uncertainty, fuzzy differential equations have been introduced by using Hukuhara difference (H-difference) and the corresponding Hukukara differentiability (H-derivative) in [17]. Thereafter, a lot of works are dedicated to the investigation of the existence and uniqueness of fuzzy differential equations under different conditions [19]. However, this approach suffers from a serious disadvantage that the length of the support or the diameter of the solution is increasing as the time increases [9]. Therefore, it causes an obstacle to discuss some behavior, such as stability, periodicity, etc. To overcome this disadvantage, a fuzzy differential equation is interpreted as a system of differential inclusions [11]. Under this improved method, some related problems associated with fuzzy differential equations, such as Lyapunov stability, periodicity of the fuzzy solution, are considered [9]. In this case, the main shortcoming using differential inclusions is that there is no derivative of a fuzzy number-valued function (mapping). Later, Bede and Gal [4,5] have proposed the concept of strongly generalized differentiability of a fuzzy number-valued mapping by using H-difference, which enlarged the class of the differentiable fuzzy number-valued mappings introduced by Puri and Ralescu [30]. To a certain extent, this concept overcomes these shortcomings mentioned above.

First order linear fuzzy differential equations are one of the most basic fuzzy differential equations which may appear in many applications. Although the form of this type of equation is very simple, the structure and behavior exhibit different characteristics under different representations and interpretations. Until now, the solutions of first order linear fuzzy differential equations have been extensively studied by various authors from different perspectives [6,18,26,27].

As we all know, for a first order linear differential equation, neither the form of the equation nor the sign of the coefficient will affect its solution. More precisely, each item can be moved freely from both sides of the equation, and meantime, the solution is also independent of the sign of the coefficient of unknown function. However, the situation is more complicated in a fuzzy setting. The relevant results obtained in [6,18,26] have shown that the formal solution of a first order linear fuzzy differential equation is not only related to the form of the equation, but also depends on the sign of the coefficient bearing on the unknown function and the definition of differentiability. From the existing results, the first order linear fuzzy differential equations are summarized in the following three forms: