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Short communication

A note on the generalized difference and the generalized differentiability

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Abstract

This note reveals that the assertion in Stefanini (2008) [2] that the generalized difference between two fuzzy numbers is always a fuzzy number is incorrect. As a consequence, the theorem of existence of the generalized differentiability in Bede and Stefanini (2013) [7] is also incorrect. We propose a modification in the definition of the generalized difference in order to assure the mentioned results.

Keywords: Generalized difference; Generalized derivative; Fuzzy arithmetic; Fuzzy numbers

1. Introduction

The Hukuhara difference ($\ominus_H$) for fuzzy numbers avoids a certain property of the difference based on standard interval arithmetic. Using the latter, the difference between two non-crisp numbers is always a non-crisp number. Hence given a fuzzy number A it is not possible that $A - A = \{0\}$. The Hukuhara difference, based on the Hukuhara difference for intervals [1], allows that the difference between two non-crisp quantities be a crisp value and presents the desired property $A \ominus_H A = \{0\}$. But this difference is not defined for every pair of fuzzy numbers. In fact, a necessary condition for the Hukuhara difference $A \ominus_H B$ between the fuzzy numbers A and B to exist is that the diameters of the α-cuts of A are bigger than the diameters of the corresponding α-cuts of B. In order to have a difference with the mentioned desired property and defined for a wider class of pairs of fuzzy numbers, Stefanini [2,3] proposed the generalized Hukuhara difference. However, there are pairs of fuzzy numbers whose generalized Hukuhara difference does not define a fuzzy number. In the same studies Stefanini [2,3] generalized the previous difference and proposed the generalized difference (also called "approximated fuzzy gH-difference" in [3]) and asserted that the difference between two fuzzy numbers is always a fuzzy number. However, [4] presented a counter-example to this affirmation recently.

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The Hukuhara and the generalized differences are used to define differentiability of fuzzy-number-valued functions [5–7]. The relation among the resulting derivatives reflects the relation among the differences. That is, the generalized Hukuhara derivative is defined for more cases of fuzzy-number-valued functions (is more general) than the Hukuhara derivative. Moreover, the generalized derivative is the most general of these three derivatives.

The remainder of this note is organized as follows. Section 2 reviews basic concepts and presents the notation used in this text. Sections 3 presents definitions and a counter-example to results regarding the generalized difference and the generalized differentiability and suggests a modification in the definition of the generalized difference. Section 4 regards some concluding remarks.

2. Preliminaries

We denote by $\mathcal{F}(\mathbb{R})$ the family of fuzzy sets of $\mathbb{R}$ and $\mu_A : \mathbb{R} \to [0, 1]$ the membership functions of a fuzzy set A. By $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ we mean the family of fuzzy numbers, that is, the fuzzy sets of $\mathbb{R}$ whose α-cuts

$$[A]_\alpha = \begin{cases} \{x \in \mathbb{R} : \mu_A(x) \geq \alpha\}, & \text{if } \alpha > 0 \\ cl\{x \in \mathbb{R} : \mu_A(x) > 0\}, & \text{if } \alpha = 0 \end{cases}$$

are nonempty compact convex sets of $\mathbb{R}$. Each α-cut of a fuzzy number is represented by its lower and upper endpoints: $[A]_\alpha = [a_\alpha^-, a_\alpha^+]$.

The sum between two fuzzy numbers A and B is defined levelwise using standard interval arithmetic:

$$[A + B]_\alpha = [A]_\alpha + [B]_\alpha = \{a + b, a \in [A]_\alpha, b \in [B]_\alpha\}$$

for all $\alpha \in [0, 1]$.

The standard difference is also defined levelwise using standard interval arithmetic:

$$[A - B]_\alpha = [A]_\alpha - [B]_\alpha = \{a - b, a \in [A]_\alpha, b \in [B]_\alpha\}$$

for all $\alpha \in [0, 1]$.

A fuzzy-number-valued function is a function $F : I \to \mathcal{F}_{\mathcal{C}}(\mathbb{R})$ that maps the interval I into $\mathcal{F}_{\mathcal{C}}(\mathbb{R})$ and we denote $[F(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)]$. We call f_α^- and f_α^+ the level set functions of F.

3. Generalized difference and generalized differentiability

We first define the generalized difference and present the result from [3] which we show that is incorrect by a counter-example. The concept of generalized Hukuhara difference is needed to define the generalized difference.

Definition 3.1. (See [2,3].) Given two fuzzy numbers A and B, the generalized Hukuhara difference (gH-difference for short) $A \ominus_{gH} B$ is defined as the fuzzy number C such that (i) $A = B + C$ or (ii) $A - C = B$, if C exists.

Case (i) of Definition 3.1 coincides with the concept of Hukuhara difference. If A and B are two closed intervals the gH-difference is given by

$$A \ominus_{gH} B = [\min\{a^- - b^-, a^+ - b^+\}, \max\{a^- - b^-, a^+ - b^+\}].$$

Definition 3.2. (See [2,3].) The generalized difference (g-difference for short) $A \ominus_g B$ is defined levelwise by

$$[A \ominus_g B]_\alpha = cl \bigcup_{\beta \geq \alpha} \left([A]_\beta \ominus_{gH} [B]_\beta\right) \tag{1}$$

for all $\alpha \in [0, 1]$.

Stefanini in [2,3] argues that the generalized difference of any pair of fuzzy numbers is always a fuzzy number. We display a counter-example to this assertion next.