Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 238 (2014) 71–88

FUZZY sets and systems

www.elsevier.com/locate/fss

Enriched lattice-valued convergence groups ☆

T.M.G. Ahsanullah [a,*], David Gauld [b], Jawaher Al-Mufarrij [c], Fawzi Al-Thukair [a]

[a] *Department of Mathematics, King Saud University, Riyadh 11451, Saudi Arabia*
[b] *Department of Mathematics, Auckland University, New Zealand*
[c] *Department of Mathematics, Women Section, King Saud University, Riyadh 11495, Saudi Arabia*

Received 13 February 2013; received in revised form 19 August 2013; accepted 19 August 2013

Available online 27 August 2013

Abstract

Considering a category **S***L*-**GConv**, of stratified enriched *cl*-premonoid-valued generalized convergence spaces, we present the category **S***L*-**GConvGrp**, of stratified *L*-generalized convergence groups, and some of its subcategories. We present two natural examples on stratified enriched *cl*-premonoid-valued convergence groups. Among other results, we show that every stratified strong *L*-limit group is **S***L*-**UCS**-uniformizable.

Keywords: Topology; Algebra; *L*-filter; *L*-generalized convergence group; *L*-limit group; Strong *L*-limit group; Ultra approach limit group; Probabilistic limit group; *t*-norm; *L*-uniform convergence space; Enriched lattice

1. Introduction

R. Lowen introduced the concept of fuzzy convergence spaces based on the idea of prefilters [26], and developed various theories in the realm of fuzzy topological spaces [26,27] (or, stratified [0, 1]-topological spaces in the sense of [14]). In 1992, E. Lowen, R. Lowen and P. Wuyts [27] observed that the category of fuzzy topological spaces, **FTop** and the category of fuzzy uniform spaces, **FUnif** are not cartesian closed, and therefore contain various deficiencies. They proved that the category of fuzzy convergence spaces is cartesian closed and beyond [1]; i.e., for instance, it has a function space structure. U. Höhle pointed out that for an arbitrary lattice L, the notion of L-filter provides a better framework for the development of lattice-valued convergence theory [14]. Based on this idea, with L as a complete Heyting algebra, G. Jäger [16] proposed a category of L-fuzzy convergence spaces, which is cartesian closed, and possesses various elegant features [16–21]. Some other authors, notably, H. Boustique, R.N. Mohapatra and G. Richardson [6]; P.V. Flores, R.N. Mohapatra and G. Richardson [10] also use frame as lattices and tried to improve this notion by imposing certain conditions on the underlying lattices. M. Demirci [8] also made progress in achieving results on enriched lattice-valued convergence structure within the framework of fixed-basis fuzzy topological spaces in the line of [14]. P. Eklund and W. Gähler [9] made significant contributions towards fuzzy filter

☆ A part of this research was carried out while the first author was visiting the Department of Mathematics, Auckland University, New Zealand.

* Corresponding author. Tel.: +966 11 4675177; fax: +966 11 4676512.
E-mail addresses: tmga1@ksu.edu.sa (T.M.G. Ahsanullah), d.gauld@auckland.ac.nz (D. Gauld), jmufarrij@ksu.edu.sa (J. Al-Mufarrij), thukair@ksu.edu.sa (F. Al-Thukair).

0165-0114/$ – see front matter
http://dx.doi.org/10.1016/j.fss.2013.08.010

functors and convergence. G. Preuss generalized many of his classical results from [35] into the context of fuzzy convergence structures [9,36]. Considering enriched *cl*-premonoid as an underlying lattice, in 2009, A. Craig and G. Jäger [7] entered into the deeper part of convergence structures obtaining various interesting results. In 2012, D. Orpen and G. Jäger [34] generalized further various notions of convergence from complete Heyting algebra to enriched *cl*-premonoid capturing many of their previously achieved results [16,17], and also, captured many results from [38].

Following Jäger's notion of a frame-valued convergence approach [16], T.M.G. Ahsanullah and J. Al-Mufarrij [2] introduced a notion of a lattice-valued convergence group, and Ahsanullah [3] introduced a notion of a lattice-valued convergence ring, showing that every stratified frame-valued convergence ring carries in a natural way a stratified uniform convergence structure. The motivation behind this article is to generalize further our previous works on frame-valued cases to enriched *cl*-premonoid-valued cases in the line of [5,7,11,14,34]; and also, to provide some results of lattice-valued generalized convergence groups based on various concepts introduced in [17,18], and discuss the uniformizability of stratified strong L-limit group. We may observe that ultra approach limit group [4,30] can be viewed as a natural example for a stratified strong $L = ([0,1], \leqslant, \wedge, *)$-valued limit group, and probabilistic limit group can be viewed as a natural example of a stratified $L = ([0,1], \leqslant, \wedge, *)$-valued limit group, where $*$ is a triangular norm [24] or a t-norm, meaning a binary operation on $[0,1]$, which is associative, commutative, non-decreasing in each argument and which has 1 as the unit element.

Primarily, our lattice under consideration is a fixed-basis lattice $L = (L, \leqslant, \otimes, *)$ which is an enriched *cl*-premonoid with $\otimes$ as a *cl*-premonoid operation and $*$ as a $\mathbf{G}L$-monoid operation; however, for some technical difficulties in dealing with product L-filters, some times we need to change our lattice structure to $L = (L, \leqslant, \otimes = *, *)$ as our enriched lattice. In Section 3, we introduce various notions of lattice-valued convergence groups, and study some basic properties, and in Section 4, we provide two natural examples (Propositions 4.5 and 4.9), first one is based on the idea of approach convergence spaces attributed to E. Lowen and R. Lowen [28] and, the other is based on the probabilistic limit spaces under t-norm due to H. Herrlich and D. Zhang [12] (see also [22,32,34]). We also present an example (Proposition 4.11) of a stratified framed-valued generalized convergence group based on function space structure in the category of frame-valued convergence spaces [16]. We construct in Section 5, the left and right L-uniform convergence structures based on noncommutative groups. Here we are able to show that every stratified strong L-limit group carries in a natural way a stratified L-uniform convergence structure. We refer to [23] and [31] for the classical results on convergence groups.

2. Preliminaries

Throughout the text we consider $L = (L, \leqslant)$ a complete lattice with $\top$, the top element and $\bot$, the bottom element of L.

Definition 2.1. (See [11,14].) A triple $(L, \leqslant, *)$, where $* : L \times L \to L$ is a binary operation on L, is called a $\mathbf{G}L$-monoid if and only if the following hold:

(GLM1) $(L, *)$ is a commutative semigroup;
(GLM2) $\forall \alpha \in L$: $\alpha * \top = \alpha$;
(GLM3) $*$ is distributive over arbitrary joins:

$$\gamma * \left(\bigvee_{k \in K} \alpha_k \right) = \bigvee_{k \in K} (\gamma * \alpha_k), \qquad \text{for } k \in K,\ \alpha_k, \gamma \in L;$$

(GLM4) for every $\gamma \leqslant \alpha$ there exists $\beta \in L$ such that $\gamma = \alpha * \beta$ (divisibility).

The triple $(L, \leqslant, *)$ is called a commutative quantale if (GLM1)–(GLM3) are fulfilled. If $* = \wedge$, then the triple $(L, \leqslant, \wedge)$ is called a frame or a complete Heyting algebra [37].

For a commutative quantale, the implication operator $\to$, also known as residuum, is given by

$$\to : L \times L \to L, \qquad \alpha \to \beta = \bigvee \{\gamma \in L \mid \alpha * \gamma \leqslant \beta\}.$$