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Fuzzy Sets and Systems 238 (2014) 102–112

FUZZY sets and systems

www.elsevier.com/locate/fss

Two cartesian closed subcategories of fuzzy domains ☆

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Received 25 September 2012; received in revised form 9 July 2013; accepted 16 July 2013

Available online 24 July 2013

Abstract

The aim of this paper is to search for cartesian closed subcategories of fuzzy domains. Based on the notion of retraction–embedding pair in the category of fuzzy dcpos, we prove that the category of fuzzy continuous lattices and the category of fuzzy algebraic lattices are cartesian closed.

Keywords: Fuzzy relations; Category theory; Quantitative domain; Fuzzy poset; Fuzzy domain; Cartesian closed category

1. Introduction

Domain theory [1,10] originated in work by Dana Scott [22,23] is a formal basis for the semantics of programming languages. In the early eighties, de Bakker and Zucker [3] presented a quantitative model of concurrent processes based on metric spaces, which put forward a challenge to the modelling power that domain theory can offer. Thus, since that time much effort has been spent on searching for a class of mathematical structures that can serve as (quantitative) domains of computation. Quantitative domain theory (QDT for short) [6–9,21,27], which refines ordinary domain theory by replacing the qualitative notion of approximation by a quantitative notion of degree of approximation, has undergone active research in the past three decades. There have been several approaches to QDT, including: Rutten's generalized (ultra)metric spaces [21]; Flagg's continuity spaces [8,9]; and Wagner's quantale enriched categories (Ω-categories for short) [27,28]. The Ω-category approach subsumes the former ones. And, in order to get better behavior of Ω-categories (as quantitative domains) one often has to impose further conditions on the quantale Ω. So, some authors choose to work with completely distributive quantales [12]; some with Girard quantales [29]; and so on.

In this paper, we work with frames [13,20]. That is, the quantale Ω is assumed to be a frame. In this case Ω-categories will be called fuzzy posets. Making use of the notion of ideal in Lai and Zhang [17], Yao [31,32] introduced the notion of fuzzy dcpos and proved that the category of fuzzy dcpos with fuzzy Scott continuous functions as morphisms is cartesian closed. In this paper, we are going to prove that the category of fuzzy continuous lattices and

☆ The research is supported by the National Natural Science Foundation of China (Grant Nos. 11171196, 10871121).

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0165-0114/$ – see front matter
http://dx.doi.org/10.1016/j.fss.2013.07.015

the category of fuzzy algebraic lattices are cartesian closed. These results generalize the well-known results that the categories of continuous lattices and algebraic lattices are cartesian closed to the fuzzy setting.

The contents of the paper are organized as follows. Section 2 lists some preliminary notions and results about fuzzy posets. In Section 3, we recall the concepts of fuzzy continuous and fuzzy algebraic dcpos and study the retractions of them. In Section 4, we show that the category of fuzzy algebraic lattices and the category of fuzzy continuous lattices are cartesian closed.

2. Complete and directed complete fuzzy posets

In this paper, L always denotes a *frame*. The greatest element of L is denoted 1 and the least element of L is denoted 0. For $A \subseteq L$, the least upper bound (resp., greatest lower bound) of A is written as $\bigvee A$ (resp., $\bigwedge A$). In particular, $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$. For $a, b \in L$, we define $a \to b = \bigvee\{c \in L \mid a \wedge c \leqslant b\}$. A mapping from a set X to L is called an L-*subset* of X. For two sets X and Y, Y^X denotes the set of all maps from X to Y.

Definition 2.1. (See Fan [6], Bělohlávek [4].) A *fuzzy poset* is a pair (X, e) such that X is a set and $e : L \times L \longrightarrow L$ is a map, called a *fuzzy partial order*, that satisfies for every $x, y, z \in X$,

(E1) $e(x, x) = 1$ (reflexivity);

(E2) $e(x, y) \wedge e(y, z) \leqslant e(x, z)$ (transitivity);

(E3) $e(x, y) = e(y, z) = 1$ implies $x = y$ (antisymmetry).

If there is no confusion, a fuzzy poset (X, e) will be simply denoted by X.

A map $f : (X, e_X) \longrightarrow (Y, e_Y)$ between fuzzy posets is called *monotone* if $e_X(x_1, x_2) \leqslant e_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. A monotone map is called a *fuzzy order embedding* if $e_X(x_1, x_2) = e_Y(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. A surjective fuzzy order embedding is called a *fuzzy order isomorphism*. If there is a fuzzy order isomorphism $f : X \longrightarrow Y$, then X and Y are said to be *isomorphic* to each other, in symbols $X \cong Y$. Clearly, a monotone map $f : X \longrightarrow Y$ is a fuzzy order isomorphism iff there is a monotone map $g : Y \longrightarrow X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$.

Remark 2.2. (1) The notion of fuzzy order was introduced by Zadeh [34] in the case that the truth-value table $L = ([0, 1], \wedge)$. Later, this notion was generalized by Bělohlávek [4] and Fan [6] with deferent names. In fact, when L is a frame, Fan's L-fuzzy posets are equivalent to Bělohlávek's L-ordered sets [30]. Both of them are a particular case of Ω-categories. Thus, the name fuzzy poset is used [31,37] to replace the original names.

(2) Let (X, e) be a fuzzy poset. The fuzzy partial order e induces a partial order $\leqslant_e$ on X defined by $x \leqslant_e y$ iff $e(x, y) = 1$. We will denote $\leqslant_e$ simply by $\leqslant$, and denote the join operation in $(X, \leqslant_e)$ by $\vee$. Conversely, given a poset $(X, \leqslant)$, define $e_\leqslant : X \times X \longrightarrow L$ by $e_\leqslant(x, y) = 1$ if $x \leqslant y$ and $e_\leqslant(x, y) = 0$, otherwise, then $(X, e_\leqslant)$ is a fuzzy poset. Let $\mathbf{2} = (\{0, 1\}, \wedge, 1)$, where $\{0, 1\}$ is a complete lattice with the ordering $0 < 1$. When $L = \mathbf{2}$, a fuzzy poset is just a partially ordered set.

Example 2.3. (1) (The canonical fuzzy partial order on L) Define $e_L : L \times L \longrightarrow L$ by $e_L(x, y) = x \to y$ for all $x, y \in L$. Then (L, e_L) is a fuzzy poset.

(2) (Sub-fuzzy poset) Let (X, e) be a fuzzy poset and $Y \subseteq X$. Then $(Y, e|_Y)$ is a fuzzy poset, where $e|_Y : Y \times Y \longrightarrow L$ is the restriction of e to $Y \times Y$, we will write $(Y, e|_Y)$ simply as (Y, e).

(3) (Dual fuzzy poset) Let (X, e) be a fuzzy poset, define $e^{op} : L \times L \longrightarrow L$ by $e^{op}(x, y) = e(y, x)$. Then (X, e^{op}) is a fuzzy poset, which will be denoted by X^{op}.

(4) Let X be a set. For $A, B \in L^X$, the *subsethood degree* [11] $\mathrm{sub}(A, B)$ of A in B is defined by $\mathrm{sub}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then (L^X, sub) is a fuzzy poset [4].

(5) (Discrete fuzzy poset) Given a set X and $x, y \in X$, let $d(x, y) = 1$ if $x = y$ and $d(x, y) = 0$ if $x \neq y$. Then (X, d) is a fuzzy poset. If $X = \{x\}$ is a singleton, then (X, d) is the terminal object in the category of fuzzy posets.

(6) (Product fuzzy poset) Let $\{(X_i, e_i)\}_{i \in \Lambda}$ be a non-empty family of fuzzy posets, $X = \prod_{i \in \Lambda} X_i$. Define $e : X \times X \longrightarrow L$ by $\forall x = (x_i)_{i \in \Lambda}, y = (y_i)_{i \in \Lambda} \in X$, $e(x, y) = \bigwedge_{i \in \Lambda} e_i(x_i, y_i)$. Then (X, e) is a fuzzy poset, e is said to be the *product fuzzy partial order* of $\{e_i \mid i \in \Lambda\}$.