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Pseudo-L^p space and convergence

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Abstract

In the framework of the pseudo-analysis the classical L^p space is generalized and there are proved important properties of introduced space. Three types of convergence of sequences of measurable functions are considered in this space. The inequalities for integrals based on pseudo-integral have been recently proposed as the Hölder, Minkowski and Markov inequalities, which are applied in observing relationship among introduced convergences.

Keywords: Pseudo-addition; Pseudo-multiplication; Semiring; Nonadditive measures; Pseudo-integral; Integral inequalities; Convergence

1. Introduction

Many important applications, mostly because of nonlinearity and involved uncertainty (not covered by the probability theory) of the considered problems, require the use of different operations than usual addition and multiplication of reals. Therefore in the approach of pseudo-analysis the field of real numbers is replaced with a semiring, i.e., a real interval $[a, b] \subseteq [-\infty, \infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$, see [21,29,30,32,33,35,36,44]. Pseudo-analysis includes also the theory of idempotent measure of Maslov and his collaborators, see [16,17,20]. The tools of the pseudo-analysis find applications in many different fields such as system theory, uncertain dynamical systems, optimization, decision making, control theory, differential equations, difference equations, fuzzy logics, see [13,17,20,30,31,33,43,46]. In this paper we develop in the framework of pseudo-analysis necessary generalization of the classical L^p space.

The classical measure theory is based on countable additive measures, see [41]. Although the additive measures are widely used, they do not allow modeling many phenomena involving interaction between criteria. Therefore, nonadditive measure, called also fuzzy measure, and the corresponding integrals, e.g., Choquet, Sugeno, are introduced, see [9,14,30,34,43]. The Choquet and Sugeno integrals have important applications as aggregation functions in decision theory (multiple criteria, multiple attributes, multiperson decision making, multiobjective optimization), information or data fusion, artificial intelligence and fuzzy logics, see [12,42,45,46]. One of the proposed useful nonadditive

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measure is the pseudo-additive measure. Based on those measures the pseudo-analysis as a generalization of the classical analysis is developed. There are usually considered three important special cases of the real semirings on an interval $[a,b]$ (see Cases I, II and III in Section 2.1), which often occurs in applications. Namely, continuous pseudo-addition has a representation as ordinal sum of generated pseudo-additions, see [14,44]. From investigations of a pair of pseudo-operation, which is managed in [24], it turns out that the mentioned three cases are canonical representatives. Using this special type of semirings the notions of $\oplus$-measure (pseudo-additive measure) and corresponding integral (pseudo-integral) were introduced. Since integrals based on nonadditive measure have wide application, the inequalities for those integrals have been studied. The inequalities for Choquet and Sugeno integral were given in [1,2,4,10,11,22,23,25,26,40]. Jensen type inequality for Sugeno integral was in [40] and fuzzy Chebyshev type inequality has been obtained by a several authors, see [2,10,23,25,27]. Inequalities with respect to the Choquet integral are observed in [47] and [22]. The generalizations of the classical integral inequalities for the universal integral (introduced in [15]) were investigated in [6]. In [3,5,37–39] inequalities with respect to pseudo-integrals were considered. Hölder, Minkowski and Markov inequalities for Lebesgue integral, see [41], play an important role in many areas of mathematics, especially in probability theory, information sciences, economics, engineering. Several different notions of convergence of sequences of measurable functions and relationship between such convergence concepts are a very important part of the classical measure theory. All those types of convergence have applications in the probability theory, differential equations, applied mathematics, stochastic processes, etc., see [34].

Several convergence concepts based on the Sugeno and Choquet integrals are observed, respectively, in [46] and [47]. In the framework of the theory of idempotent measure of Maslov an analog of L^p space and the convergence of decision variables were presented in [8]. Using the tools of the pseudo-analysis, a generalization of the classical L^p space is constructed in [7].

In Section 2 of this paper we recall the notions of pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ forming a real semiring on the interval $[a,b]$. Then we recall the notion of $\oplus$-measure and corresponding pseudo-integral based on this measure. First, in Section 3 is defined a general notion of the pseudo-metric on an arbitrary set, and then it is introduced a generalization of the classical L^p space of functions, where we define a corresponding pseudo-metric. Using the previously proved Hölder, Minkowski and Markov type inequalities for the pseudo-integral, there are obtained important properties of the pseudo-L^p space.

The several types of convergence are introduces in Section 5. The relationships among these types of convergence are given for the semiring with generated pseudo-operations in Section 5.1 and for the semiring with idempotent pseudo-addition in Section 5.2.

2. Pseudo-integral

2.1. Pseudo-operations

Let $[a,b]$ be a closed subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preccurlyeq$. This can be the usual order of the real line, but it can be another order. The operation $\oplus$ (pseudo-addition) is a commutative, non-decreasing (with respect to $\preccurlyeq$), associative function $\oplus : [a,b]\times[a,b]\to[a,b]$ with a zero (neutral) element denoted by $\mathbf{0}$. Denote $[a,b]_+ = \{x \mid x\in[a,b],\ \mathbf{0}\preccurlyeq x\}$. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b]\times[a,b]\to[a,b]$ which is commutative, positively non-decreasing, i.e., $x\preccurlyeq y$ implies $x\odot z\preccurlyeq y\odot z$, $z\in[a,b]_+$, associative and for which there exists a unit element $\mathbf{1}\in[a,b]$, i.e., for each $x\in[a,b]$, $1\odot x = x$. We assume $\mathbf{0}\odot x=\mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x\odot(y\oplus z) = (x\odot y)\oplus(x\odot z).$$

The structure $([a,b],\oplus,\odot)$ is called a *real semiring on the interval* $[a,b]$ (see [18,30]). We will introduce many notions for the general real semiring on the interval $[a,b]$, but most of the obtained results will be given for the following three important real semirings on the interval with continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x\oplus y = \sup(x,y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a,b]$. We have $\mathbf{0}=a$ and the idempotent operation sup induces a full order in the following way: $x\preccurlyeq y$ if and only if $\sup(x,y)=y$.