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A note on compactness in a fuzzy metric space

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Abstract

We study the compactness in a fuzzy metric space. We also give a generalization of Niemytzki–Tychonoff theorem for a fuzzy metric space.

Keywords: Fuzzy metric space; Compactness; Niemytzki–Tychonoff theorem

1. Introduction

In the literature there are several notions of a fuzzy metric space. The first one was introduced by Kramosil and Michálek [13]. Its motivation derives from a statistical metric space. As Theorem 1 in [13] states: *Any fuzzy metric space is equivalent to a statistical metric space*. Later the notion was modified by George and Veeramani [2]. Various aspects of this kind of fuzzy metric space was studied among others by George and Veeramani [3], Gregori and Romaguera [4,5] and Miheţ [15].

The second notion of a fuzzy metric space was introduced by Kaleva and Seikkala in [10]. The idea behind this notion was to fuzzify the classical metric. For the properties of this fuzzy metric space see for instance Fang [1], Hadžić and Pap [6], Huang and Wu [7], Jung et al. [8], Kaleva [9] and Xiao et al. [17].

Since the fuzzy metric space by Kaleva and Seikkala just replaces real values of a metric by fuzzy values and triangle inequality is modified for this setting, it is expected that in many ways the fuzzy metric space resembles the ordinary metric space. Compactness is a central notion in topology. In this paper we study the compactness in a fuzzy metric space.

2. Preliminaries

Denote $\mathcal{F}^1 = \{u : \mathbb{R} \to [0, 1] \mid u \text{ satisfies (i)–(iv) below}\}$, where

(i) u is normal, i.e. there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
(ii) u is quasiconcave,

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(iii) u is upper semicontinuous,
(iv) u is compactly supported.

The elements of $\mathcal{F}^1$ are called fuzzy numbers. It is well known that the α-level sets of a fuzzy number u are non-empty, compact intervals in $\mathbb{R}$, denoted by $[u]_\alpha = [\lambda_\alpha(u), \rho_\alpha(u)]$, $0 \leqslant \alpha \leqslant 1$. A fuzzy number g is non-negative if $g(t) = 0$ for all $t < 0$. The collection of non-negative fuzzy numbers is denoted by $\mathcal{G}$.

We assume that $\mathbb{R}$ is endowed with the usual topology. A sequence $\{u_n\}$ in $\mathcal{F}^1$ converges levelwise to $u \in \mathcal{F}^1$, denoted $\lim_{n\to\infty} u_n = u$, if for all $\alpha \in [0, 1]$

$$\lim_{n\to\infty} \lambda_\alpha(u_n) = \lambda_\alpha(u) \quad \text{and} \quad \lim_{n\to\infty} \rho_\alpha(u_n) = \rho_\alpha(u).$$

If $\{u_n\} \subset \mathcal{G}$ and $u = \bar{0}$, defined by $\bar{0}(t) = \begin{cases} 1, & \text{if } t = 0, \\ 0, & \text{elsewhere,} \end{cases}$ then for all $\alpha \in [0, 1]$ we have

$$0 = \lambda_\alpha(u) = \rho_\alpha(u) \leqslant \lambda_\alpha(u_n) \leqslant \rho_\alpha(u_n) \leqslant \rho_0(u_n).$$

Hence $\lim_{n\to\infty} u_n = \bar{0}$ if and only if $\lim_{n\to\infty} \rho_0(u_n) = 0$.

In [10] Kaleva and Seikkala introduced a fuzzy metric space as follows.

Definition 2.1. Let X be a set and $L, R: [0, 1] \times [0, 1] \to [0, 1]$ be two symmetric, non-decreasing mappings in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$. A function $d: X \times X \to \mathcal{G}$ is called a fuzzy metric if

(i) $d(x, y) = \bar{0}$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) for all $x, y, z \in X$,
  (a) $d(x, y)(s + t) \geqslant L(d(x, z)(s), d(z, y)(t))$ whenever $s \leqslant \lambda_1(x, z)$, $t \leqslant \lambda_1(z, y)$ and $s + t \leqslant \lambda_1(x, y)$,
  (b) $d(x, y)(s + t) \leqslant R(d(x, z)(s), d(z, y)(t))$ whenever $s \geqslant \lambda_1(x, z)$, $t \geqslant \lambda_1(z, y)$ and $s + t \geqslant \lambda_1(x, y)$,

where

$$\big[d(x, y)\big]_\alpha = \big[\lambda_\alpha(x, y), \rho_\alpha(x, y)\big] \quad \text{for all } x, y \in X,\ 0 \leqslant \alpha \leqslant 1,$$

denotes the α-level interval of $d(x, y)$. The quadruple (X, d, L, R) is called a fuzzy metric space.

In the sequel we assume that all fuzzy metric spaces (X, d, L, R) also satisfy the condition $\lim_{a\to 0_+} R(a, a) = 0$. Since R is non-decreasing, it follows that $R(0, 0) = 0$.

Example 2.1. Let (X, δ) be a metric space and define $d: X \times X \to \mathcal{G}$ by

$$d(x, y)(t) = \begin{cases} 1, & t = \delta(x, y), \\ 0, & \text{elsewhere.} \end{cases}$$

Since

$$\lambda_\alpha(x, y) = \delta(x, y) = \rho_\alpha(x, y) \quad \text{for all } x, y \in X,\ \alpha \in [0, 1],$$

then since δ satisfies the triangle inequality we have for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in [0, 1]$

$$\lambda_\gamma(x, y) \leqslant \lambda_\alpha(x, z) + \lambda_\beta(z, y) \quad \text{and} \quad \rho_\gamma(x, y) \leqslant \rho_\alpha(x, z) + \rho_\beta(z, y).$$

If $L(0, 1) = 0$ and R is right continuous, then by Theorems 3.4 and 4.5 in [7] we deduce that (X, d, L, R) is a fuzzy metric space. Hence a metric space is a special case of a fuzzy metric space.

However, if $L \geqslant \max$ then by Lemma 3.1 in [10] the inequality (iii)(a) in Definition 2.1 implies that $\lambda_1(x, y) = 0$ for all $x, y \in X$. Hence (X, d, L, R) is not a fuzzy metric space.

We also need the following definitions.