Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 278 (2015) 3–19

FUZZY sets and systems

www.elsevier.com/locate/fss

Conditional extensions of fuzzy preorders

J.C.R. Alcantud [a], S. Díaz [b,*]

[a] *Facultad de Economía y Empresa and Multidisciplinary Institute of Enterprise (IME), Campus Miguel de Unamuno, Universidad de Salamanca, Spain*

[b] *Dept. of Statistics and O. R., University of Oviedo, Calvo Sotelo s/n, 33007 Oviedo, Spain*

Received 21 April 2014; received in revised form 11 September 2014; accepted 13 September 2014

Available online 23 September 2014

Abstract

The problem of embedding incomplete into complete relations has been an important topic of research in the context of crisp relations and their applications. Several variations of the acclaimed Szpilrajn's theorem have been provided, inclusive of the case when some order conditions between elements are imposed on the extension. We extend the analysis of that topic by Alcantud to the fuzzy case. By appealing to generators to decompose (fuzzy) preference relations into strict preference and indifference relations, we give general extension results for the corresponding concept of compatible extension of a fuzzy reflexive relation. Then we investigate the conditions under which compatible order extensions exist such that certain elements are connected by the asymmetric part, resp., and certain other elements by the symmetric part, to respective elements with degree 1.

Keywords: Fuzzy relation; Transitivity; Acyclicity; Consistency; Extension; (Indifference) generator

1. Introduction

As Herden and Pallack [27] put it, "one of the best known theorems in order theory, mathematical logic, computer sciences and mathematical social sciences is the Szpilrajn Theorem which states that every partial order can be refined to a linear order". Another form of the same principle states that any preorder has an order extension (cf., Arrow [2, Chapter VI], Hansson [26, Lemma 3]). Many variations and generalizations followed. Dushnik and Miller [17] prove that any partial order is the intersection of linear orders, which is obtained as a special case of a general extension theorem by Duggan [16]. Donaldson and Weymark [14] (see also Bossert [7]) prove the corresponding result for preorders, namely, that any preorder is the intersection of orders. Suzumura [41], [42, Theorem A(5)] shows that a property called consistency is necessary and sufficient for the existence of an order extension. Alcantud [1] systematizes the identification of constraints that can be imposed on the order extensions. This is important because Szpilrajn's theorem [44] is not constructive thus the researcher cannot proceed by direct inspection of the resulting

* Corresponding author.
E-mail addresses: jcr@usal.es (J.C.R. Alcantud), diazsusana@uniovi.es (S. Díaz).
URLs: http://diarium.usal.es/jcr (J.C.R. Alcantud), http://unimode.uniovi.es/ (S. Díaz).

http://dx.doi.org/10.1016/j.fss.2014.09.009

orders.[1] Another general scheme for extending Szpilrajn's theorem consists of restricting attention to preorders with some prescribed property, and requiring that their extensions also possess that property. In this line of inspection Foldes and Szigeti [21] give a complete description of the so-called maximal compatible partial orders on A of the arbitrary unary operation $f : A \to A$, and Mabrouk [33] proves that for any preorder on a real vector space that verifies the property of translation invariance, there is an order that extends it and verifies translation invariance. Bosi and Herden [5,6], Bossert et al. [8], Campión et al. [11], Herden and Pallack [27], Jaffray [29], or Yi [46] among others discuss continuity and semicontinuity issues in relation with the Szpilrajn theorem. Particularly fruitful is Yi's approach. According to Campión et al. [11], a topological space verifies Yi's extension property – which is more restrictive than topological normality as a separation axiom – when any closed preorder on any closed subset of the space has a continuous extension to the whole space. These authors prove that even a stronger version of its semicontinuous analogue is true irrespective of the topology.

Results of Szpilrajn type have been successfully applied to many branches. In economics, they play an important role in: (*i*) revealed preference theory (cf., Richter [38]), (*ii*) the theory of choice (e.g., Chambers and Yenmez [13] use the Szpilrajn theorem to characterize responsive choice functions, that have been extensively used in matching theory and real matching markets) and social choice (e.g., in the analysis of social decision rules, the characterization of ordinal relations by Sholomov [40]; in the analysis of voting rules, Laruelle and Valenciano [32, Section 5] benefit from the Dushnik-Miller approach), see also Lahiri [31], Nehring and Puppe [35], Suzumura [41,42], Weymark [45] as a sample; (*iii*) aggregation of infinite utility streams, a field where one of its milestones, namely, Svensson's theorem [43], is a direct application of the Szpilrajn theorem; also in this field, Mabrouk [33, Section 4] obtains yet another consequence from his aforementioned result; or (*iv*) welfare economics, where Donaldson and Weymark [14] applied their extension theorem in a variety of examples. Further applications include game theory (cf., e.g., Bade [3]), computability theory (e.g., Roy [39] states the recursive version of the Szpilrajn theorem), order theory (Downey [15]), ... We address the reader to Bosi and Herden [6] for a more extensive list of applications.

In this fruitful field of research we can also name various fuzzy versions or extensions of Szpilrajn's theorem. Among them, Georgescu [22, Theorem 5.4], [23, Corollary 4.37] and [24, Theorem 4.17], Bodenhofer and Klawonn [4, Theorem 6.7] – who conduct a detailed investigation of linearity axioms for fuzzy orders –, Gottwald [25, Proposition 2.34], Höhle and Blanchard [28] – who produce variations for *antisymmetric* preorders both of the extension theorem in their Theorem II.7 and of the intersection theorem in their Corollary II.8 –, or Zadeh [48, Theorem 8].

In this paper we contribute to the field by providing several new variations of the Szpilrajn theorem for fuzzy preorders. We proceed in the spirit of Alcantud [1], thus we are concerned with the identification of constraints that can be imposed to the fuzzy order extensions of fuzzy relations. For the purpose of defining the key concept of compatible extension of a fuzzy reflexive relation, we consider the successful approach to the construction of an asymmetric part from the original relation that is based on generators (cf., De Baets and Fodor [9]). Firstly we give general extending results that reproduce useful statements in Georgescu [22] under the alternative construction here adopted. As an application, the recourse to 1-Lipschitz generators allows us to identify precisely which lists of pairs of connections with degree 1 by the asymmetric part can be imposed to the compatible extensions of a fuzzy preorder. The general case of imposing pairs of connections with degree 1 by either the asymmetric or the symmetric part is investigated under the generator min. This restriction permits to prove a sufficient condition that is not necessary. We also check that the only generator for which our proposed condition is sufficient is the minimum.

This paper is organized as follows. In Section 2 we recall the results by Alcantud [1] that inspire our investigation. We prove slightly more general versions of his two main theorems, stated for a wider class of binary relations. In Section 3 we give some notation and preliminaries on fuzzy relations and their extensions. Section 4 solves the problems posed. We briefly recall the main facts about (indifference) generators. Then we prove our results on unrestricted extensions in Section 4.1, and we apply them to tackle the aforementioned conditional extension problems in Section 4.2. Theorem 4 gives a precise answer for a particular specification of the problem under fairly general conditions, and Theorem 5 and the discussion afterwards concern the general case. In Section 5 we conclude and pose questions for further research.

[1] The case of finite sets is different. In such case, extensions *à la Szpilrajn* can be obtained in a somewhat constructive way, using classical techniques like "labelling": see, e.g., Caspard et al. [12].