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Preassociative aggregation functions

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Abstract

The classical property of associativity is very often considered in aggregation function theory and fuzzy logic. In this paper we provide axiomatizations of various classes of preassociative functions, where preassociativity is a generalization of associativity recently introduced by the authors. These axiomatizations are based on existing characterizations of some noteworthy classes of associative operations, such as the class of Aczélian semigroups and the class of t-norms.

Keywords: Aggregation; Associativity; Preassociativity; Functional equation; Axiomatization

1. Introduction

Let X be an arbitrary nonempty set (e.g., a nontrivial real interval) and let $X^* = \bigcup_{n\geqslant 0} X^n$ be the set of all tuples on X, with the convention that $X^0 = \{\varepsilon\}$ (i.e., ε denotes the unique 0-tuple on X). The *length* $|\mathbf{x}|$ of a tuple $\mathbf{x} \in X^*$ is a nonnegative integer defined in the usual way: we have $|\mathbf{x}| = n$ if and only if $\mathbf{x} \in X^n$. In particular, we have $|\varepsilon| = 0$.

In this paper we are interested in *n-ary* functions $F: X^n \to Y$, where $n \geqslant 1$ is an integer, as well as in *variadic* functions $F: X^* \to Y$, where Y is a nonempty set. A variadic function $F: X^* \to Y$ is said to be *standard* [16] if the equality $F(\mathbf{x}) = F(\varepsilon)$ holds only if $\mathbf{x} = \varepsilon$. Finally, a variadic function $F: X^* \to X \cup \{\varepsilon\}$ is called a *variadic operation on* X (or an *operation* for short), and we say that such an operation is *ε-preserving standard* (or *ε-standard* for short) if it is standard and satisfies $F(\varepsilon) = \varepsilon$.

For any variadic function $F: X^* \to Y$ and any integer $n \geqslant 0$, we denote by F_n the *n-ary part* of F, i.e., the restriction $F|_{X^n}$ of F to the set X^n. The restriction $F|_{X^*\setminus\{\varepsilon\}}$ of F to the tuples of positive lengths is denoted $F^\flat$ and called the *non-nullary part* of F. Finally, the value $F(\varepsilon)$ is called the *default value* of F.

The classical concept of associativity for binary operations can be easily generalized to variadic operations in the following way. A variadic operation $F: X^* \to X \cup \{\varepsilon\}$ is said to be *associative* [16,21] (see also [18, p. 24]) if

$$F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = F\big(\mathbf{x}, F(\mathbf{y}), \mathbf{z}\big), \quad \mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*. \tag{1}$$

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Here and throughout, for tuples $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_m)$ in X^*, the notation $F(\mathbf{x}, \mathbf{y})$ stands for the function $F(x_1, \dots, x_n, y_1, \dots, y_m)$, and similarly for more than two tuples. We also assume that $F(\varepsilon, \mathbf{x}) = F(\mathbf{x}, \varepsilon) = F(\mathbf{x})$ for every $\mathbf{x} \in X^*$.

Any associative operation $F: X^* \to X \cup \{\varepsilon\}$ clearly satisfies the condition $F(\varepsilon) = F(F(\varepsilon))$. From this observation it follows immediately that any associative standard operation $F: X^* \to X \cup \{\varepsilon\}$ is necessarily ε-standard.

Associative binary operations and associative variadic operations are widely investigated in aggregation function theory, mainly due to the many applications in fuzzy logic (for general background, see [13]).

Associative ε-standard operations $F: X^* \to X \cup \{\varepsilon\}$ are closely related to associative binary operations $G: X^2 \to X$, which are defined as the solutions of the functional equation

$$G\big(G(x, y), z\big) = G\big(x, G(y, z)\big), \quad x, y, z \in X.$$

In fact, it can be easily seen [21,22] that a binary operation $G: X^2 \to X$ is associative if and only if there exists an associative ε-standard operation $F: X^* \to X \cup \{\varepsilon\}$ such that $G = F_2$. Moreover, as observed in [18, p. 25] (see also [5, p. 15] and [13, p. 33]), any associative ε-standard operation $F: X^* \to X \cup \{\varepsilon\}$ is completely determined by its unary and binary parts. Indeed, by associativity we have

$$F_n(x_1, \dots, x_n) = F_2\big(F_{n-1}(x_1, \dots, x_{n-1}), x_n\big), \quad n \geqslant 3, \tag{2}$$

or equivalently,

$$F_n(x_1, \dots, x_n) = F_2\big(F_2\big(\dots F_2\big(F_2(x_1, x_2), x_3\big), \dots\big), x_n\big), \quad n \geqslant 3. \tag{3}$$

In this paper we are interested in the following generalization of associativity recently introduced by the authors in [21,22] (see also [16]).

Definition 1.1. (See [21,22].) A function $F: X^* \to Y$ is said to be *preassociative* if for every $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$ we have

$$F(\mathbf{y}) = F\big(\mathbf{y}'\big) \quad \Rightarrow \quad F(\mathbf{x}, \mathbf{y}, \mathbf{z}) = F\big(\mathbf{x}, \mathbf{y}', \mathbf{z}\big).$$

We can easily observe that any ε-standard operation $F: \mathbb{R}^* \to \mathbb{R} \cup \{\varepsilon\}$ defined by $F_n(\mathbf{x}) = f(\sum_{i=1}^n x_i)$ for every integer $n \geqslant 1$, where $f: \mathbb{R} \to \mathbb{R}$ is a one-to-one function, is an example of preassociative function.

It is immediate to see that any associative ε-standard operation $F: X^* \to X \cup \{\varepsilon\}$ necessarily satisfies the equation $F_1 \circ F^\flat = F^\flat$ (take $\mathbf{x} = \mathbf{z} = \varepsilon$ in Eq. (1)) and it can be shown (Proposition 3.3) that an ε-standard operation $F: X^* \to X \cup \{\varepsilon\}$ is associative if and only if it is preassociative and satisfies $F_1 \circ F^\flat = F^\flat$.

It is noteworthy that, contrary to associativity, preassociativity does not involve any composition of functions and hence allows us to consider a codomain Y that may differ from $X \cup \{\varepsilon\}$. For instance, the length function $F: X^* \to \mathbb{R}$, defined by $F(\mathbf{x}) = |\mathbf{x}|$, is standard and preassociative.

In this paper we mainly consider preassociative standard functions $F: X^* \to Y$ for which F_1 and $F^\flat$ have the same range. (For ε-standard operations, the latter condition is an immediate consequence of the condition $F_1 \circ F^\flat = F^\flat$ and hence these preassociative functions include all the associative ε-standard operations.) In Section 3 we recall the characterization of these functions as compositions of the form $F^\flat = f \circ H^\flat$, where $H: X^* \to X \cup \{\varepsilon\}$ is an associative ε-standard operation and $f: H(X^* \setminus \{\varepsilon\}) \to Y$ is one-to-one.

In Section 4 we investigate the special case of standard functions whose unary parts are one-to-one. It turns out that this latter condition greatly simplifies the general results on associative and preassociative standard functions obtained in [21,22]. Section 5 contains the main results of this paper. We first recall axiomatizations of some noteworthy classes of associative ε-standard operations, such as the class of variadic extensions of Aczélian semigroups, the class of variadic extensions of t-norms and t-conorms, and the class of associative and range-idempotent ε-standard operations. Then we show how these axiomatizations can be extended to classes of preassociative standard functions. Finally, we address some open questions in Section 6.

Throughout the paper we make use of the following notation and terminology. We denote by $\mathbb{N}$ the set $\{1, 2, 3, \dots\}$ of strictly positive integers. The domain and range of any function f are denoted by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively. The identity operation on X is the function $\mathrm{id}: X \to X$ defined by $\mathrm{id}(x) = x$.