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A characterization of discrete uninorms having smooth underlying operators

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Abstract

In this paper discrete uninorms U such that their underlying t-norm T and t-conorm S are smooth are characterized. The different cases combining when T is the minimum or the Łukasiewicz t-norm and S is the maximum or the Łukasiewicz t-conorm, are given separately and the number of discrete uninorms with these underlying operators is given. The same is done for the general case when T and/or S are smooth, but ordinal sums.

Keywords: Aggregation function; Fuzzy connectives; Discrete uninorm; Discrete t-norm; Discrete t-conorm; Smoothness condition

1. Introduction

From a mathematical point of view, the process of merging some data into a representative output is usually given through the so-called aggregation functions, that have been extensively investigated in last decades [1,2,12,19]. There are a lot of domains of applications of aggregation functions that vary from decision making and subjective evaluations, to optimization and control, image analysis and pattern recognition. In all these fields, it is well known that the data to be aggregated oscillate among many different kinds of information, from quantitative to qualitative information. Moreover, many times some uncertainty is inherent to such information.

When data is qualitative, the fuzzy linguistic approach is a good tool to model the information because then, the qualitative terms used by experts are represented via linguistic variables instead of numerical values. In this approach, linguistic variables are often interpreted to take values on a totally ordered scale like this:

$$\mathcal{L} = \{Extremely\ Bad,\ Very\ Bad,\ Bad,\ Fair,\ Good,\ Very\ Good,\ Extremely\ Good\}. \tag{1}$$

In these cases, the representative finite chain $L_n = \{0, 1, \ldots, n\}$ is usually considered to model these linguistic hedges and several researchers have developed an extensive study of aggregation functions on L_n, usually called *discrete*

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aggregation functions. Dealing with discrete operators, the smoothness condition is usually considered as the discrete counterpart of continuity. In fact, in the discrete framework this property is equivalent to the divisibility property as well as to the Lipschitz condition. Thus, many classes of discrete aggregation functions with some smoothness condition have been studied and characterized. For instance, smooth discrete t-norms and t-conorms were characterized in [17,18], see also [6], uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ and nullnorms in [16], nullnorms without the commutative property in [9], idempotent discrete uninorms in [5], and weighted means in [14].

A common factor in all these mentioned works is that they adapt well known classes of aggregation functions on $[0, 1]$ to the discrete case. In the framework of $[0, 1]$, the class of uninorms has been deeply investigated because uninorms are an especial kind of binary aggregation functions that generalize both t-norms and t-conorms. They have proved to be useful in many application fields and this has led to an extensive study of uninorms from the pure theoretical point of view. One of the most interesting topics in this direction deals with the characterization of the different classes of uninorms, mainly uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ [11], idempotent uninorms [3,15,21], representable uninorms [4,11,20], uninorms continuous in the open unit square [8,13], compensatory uninorms [7], and even those uninorms with continuous underlying operators [10].

However, in the discrete case only the classes of discrete uninorms in $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ or discrete idempotent uninorms has been studied. In fact, it is well known that there are no smooth uninorms on L_n and so, the smoothness condition is only possible in some partial regions of L_n^2, like in the mentioned cases. However, the general case of discrete uninorms having smooth underlying operators has not yet investigated and this is the main goal of this work. Specifically, we will characterize all discrete uninorms that have smooth underlying t-norm and t-conorm and we will give the exact number of this kind of uninorms.

The paper is organized as follows. In Section 2 we give some preliminaries that will be used in the paper, and the main results are presented in Section 3. Since smooth t-norms (and t-conorms) are given by the minimum, the Łukasiewicz t-norm (the maximum or the Łukasiewicz t-conorm) or an ordinal sum of these two types of operators, we have divided our results in two sections. Specifically, in Section 3 we deal with the particular cases when the t-norm and the t-conorm are given by the minimum or the Łukasiewicz t-norm and the maximum or the Łukasiewicz t-conorm, devoting a subsection to each possible combination. In Section 4 the general case when both the t-norm or the t-conorm are ordinal sums is studied. Finally, Section 5 gives general conclusions.

2. Preliminaries

We suppose the reader to be familiar with some basic results on uninorms and their classes that can be found for instance in [7,11,13,15,21].

In these preliminaries we recall some known facts on uninorms defined on finite chains, that we will also refer to discrete uninorms. In these cases, the concrete scale to be used is not determinant and the only important fact is the number of elements of the scale (see [18]). Thus, given any positive integer n, we will deal from now on with the finite chain $L_n = \{0, 1, 2, \dots, n\}$. We will use indistinctly the interval notation $L_n = [0, n]$ and also the usual notations $[0, e]$ and $[e, n]$ when $e \in L_n$ for the corresponding subsets of L_n.

Definition 1. (See [11,16].) A *uninorm* on L_n is a two-place function $U : L_n^2 \to L_n$ which is associative, commutative, increasing in each place and such that there exists some element $e \in L_n$, called *neutral element*, such that $U(e, x) = x$ for all $x \in L_n$.

It is clear that the function U becomes a t-norm when $e = n$ and a t-conorm when $e = 0$. For any uninorm on L_n we have $U(n, 0) \in \{0, n\}$ and a uninorm U is called *conjunctive* when $U(n, 0) = 0$ and *disjunctive* when $U(n, 0) = n$. The structure of any discrete uninorm U on L_n with neutral element $0 < e < n$ is always as follows. It is given by a t-norm T on the interval $[0, e]$, by a t-conorm S on the interval $[e, n]$ and it takes values between the minimum and the maximum in all other cases, that is, in the region $E = [0, e[\times]e, n] \cup]e, n] \times [0, e[$.

Definition 2. (See [18].) A function $f : L_n \to L_n$ is said to be *smooth* whenever $|f(x) - f(x - 1)| \leq 1$ for all $x \in L_n$ such that $x \geq 1$.