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# Order-equivalent triangular norms

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**Abstract**

In this paper, an equivalence relation on the class of t-norms induced by a $T$-partial order is provided and discussed. The equivalence classes linked to some special t-norms are characterized as well as some properties preserved by the introduced equivalence. Defining the set of incomparable elements with respect to the $T$-partial order, this set is deeply investigated. Finally, we give an answer to a recently posed open problem.


*Keywords:* Triangular norm; Bounded lattice; $T$-partial order; Divisibility

## 1. Introduction

Triangular norms were introduced by Karl Menger in 1942 [24]. They play fundamental role in many particular fields of mathematics, for example in probabilistic metric spaces [25], or in fuzzy logic and their applications [7,17], as well as in the non-additive measure and integral theory [1,14–16].

A triangular norm (briefly t-norm) $T : [0,1]^2 \to [0,1]$ is a commutative, associative, non-decreasing operation on $[0,1]$ with neutral element 1 [13]. The four basic t-norms on $[0,1]$ are the minimum $T_M$, the product $T_P$, the Łukasiewicz t-norm $T_L$ and the drastic product $T_D$ given by, respectively, $T_M(x,y) = \min(x,y)$, $T_P(x,y) = xy$, $T_L(x,y) = \max(0, x+y-1)$ and

$$T_D(x,y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Also, t-norms on a bounded lattice $(L, \leq, 0, 1)$ are defined in similar way, and then the extremal t-norms $T_D$ as well as $T_\wedge$ on $L$ are defined similarly as $T_D$ and $T_M$ on [0, 1]. Several authors have studied operations and t-norms on bounded lattices, like in [2,3,8–12,21,22].

In [20], a natural order for semigroups was defined. Similarly, in [9], a partial order defined by means of t-norms on a bounded lattice was introduced. For any elements $x, y$ of a bounded lattice $L$

$$x \preceq_T y :\Leftrightarrow T(\ell, y) = x \quad \text{for some } \ell \in L,$$

where $T$ is a t-norm on $L$. This partial order $\preceq_T$ is called a $T$-partial order on $L$. Moreover, in [9], the authors have investigated some connections between the natural order $\leq$ on $L$ and the $T$-partial order $\preceq_T$ on $L$. Also, it was obtained that $\preceq_T$ implies the natural order $\leq$ but its converse need not be true. So, it was also shown that the structure $(L, \preceq_T, 0, 1)$ (which is trivially a bounded poset) need not be a lattice, in general. Moreover, some special cases when $(L, \preceq_T, 0, 1)$ is a lattice were shown in [9].

In the present paper, we introduce an equivalence on the class of t-norms on a bounded lattice $(L, \leq, 0, 1)$ based on the equality of the induced $T$-partial orders. The main aim is to characterize the preservation of some (algebraic) properties. The paper is organized as follows. We shortly recall some basic notions in Section 2. In Section 3, we define an equivalence on the class of t-norms on a bounded lattice $(L, \leq, 0, 1)$ and we determine the equivalence class of the infimum t-norm $T_\wedge$ and the weakest t-norm $T_D$. Thus, we obtain that, in the case of the standard real unit interval $L = [0, 1]$, all continuous t-norms are equivalent. Although, we give some examples illustrating that left-continuous t-norms need not be equivalent, in general. We also show by an example that the left-continuity of any of the t-norms in the equivalence class does not imply the left-continuity for another t-norm in the equivalence class. So, we have that our equivalence relation does not preserve left-continuity. By an example, we show that a t-norm and its $\varphi$-transform need not be in the same equivalence class. Although, we give a necessary and sufficient condition for a t-norm and its $\varphi$-transform to be in the same equivalence class. So, we deduce that an isomorphism isn't sufficient alone for getting an equivalence relation. In Section 4, defining the set of all incomparable elements with respect to $\preceq_T$, we study this set in detail. In Section 5, we give an answer to an open problem posed in [9]. Finally, some concluding remarks are added.

## 2. Notations, definitions and a review of previous results

**Definition 1.** (See [11].) Let $(L, \leq, 0, 1)$ be a bounded lattice. A *triangular norm* $T$ (briefly t-norm) is a binary operation on $L$ which is commutative, associative, monotone and has neutral element 1.

$$\text{Let } T_D(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $T_D$ is a t-norm on $L$. Since it holds that $T_D \leq T$ for any t-norm $T$ on $L$, $T_D$ is the smallest t-norm on $L$. The largest t-norm on a bounded lattice $(L, \leq, 0, 1)$ is given by $T_\wedge(x, y) = x \wedge y$.

**Definition 2.** (See [5].) A t-norm $T$ on $L$ is *divisible* if the following condition holds:

$$\forall x, y \in L \quad \text{with } x \leq y \quad \text{there is a } z \in L \quad \text{such that } x = T(y, z).$$

A basic example of a non-divisible t-norm on an arbitrary lattice $L$ (with card $L > 2$) is the weakest t-norm $T_D$. Trivially, the infimum $T_\wedge$ is divisible since $x \leq y$ is equivalent to $x \wedge y = x$.

**Definition 3.** (See [11].)

(i) A t-norm $T$ on a lattice $L$ is called $\vee$-*distributive* if

$$T(a, b_1 \vee b_2) = T(a, b_1) \vee T(a, b_2)$$

for every $a, b_1, b_2 \in L$.