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On fuzzy-valued operations and fuzzy-valued fuzzy sets

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Abstract

This paper is devoted to investigate fuzzy-valued operations and the lattice structures of the algebra of fuzzy values. The algebra of fuzzy values is proven to be a complete completely distributive lattice, and the list of all join-irreducible elements is provided. Fuzzy-valued t-norms and t-conorms are induced by arbitrary t-norms and t-conorms regardless of their continuity, respectively. Moreover, fuzzy-valued fuzzy implications are constructed from fuzzy-valued t-norms induced by left-continuous t-norms and residuated lattices are constructed on the algebra of fuzzy values. As a consequence, fuzzy-valued fuzzy rough sets and fuzzy-valued fuzzy complete lattices are proposed and investigated in fuzzy-valued fuzzy sets.

Keywords: Fuzzy values; Fuzzy-valued t-norms; Fuzzy-valued t-conorms; Fuzzy-valued fuzzy implications; Residuated lattices; Fuzzy-valued fuzzy rough sets

1. Introduction

Type-2 fuzzy sets were introduced by Zadeh [32] in 1975 as a generalization of ordinary fuzzy sets, which take fuzzy truth values mapping the unit interval to itself as truth values. As a consequence, ordinary fuzzy sets and interval-valued fuzzy sets are special cases of type-2 fuzzy sets. Moreover, it is pointed out in [26] that type-2 fuzzy sets are very useful in circumstances where there is need to handle more uncertainty than it is possible using ordinary fuzzy sets or interval-valued fuzzy sets. There are two different views of type-2 proposed in [6]. In the first point of view, type-2 fuzzy sets are fuzzy sets with ill-known truth values from the unit interval, which means that the truth values are assumed to be precise, even if they are out of our knowledge. In the second point of view, type-2 fuzzy sets are special fuzzy sets with fuzzy truth values. In this paper, we focus on the second point of view.

To investigate type-2 fuzzy sets, type-2 operations were studied in many published works, such as extended t-(co)norms [5,15,16,27], extended fuzzy implications [8,9] and extended aggregation operations [4,26] in accordance with Zadeh's extension principle. However, the continuity of those fuzzy operations including t-(co)norms, fuzzy implications and aggregation operations is required, when their extensions are closed on the algebra of convex fuzzy

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truth values. Chen and Kawase [2] proposed extended supremum and infimum on the algebra of fuzzy values by nests of closed intervals. The definitions of extended supremum and infimum on the algebra of fuzzy truth values were given in [14]. However, extended supremum and infimum are not always the least upper bound and greatest lower bound on the algebra of convex normal fuzzy truth values [14], respectively. In this paper, we study the problem when extended supremum and infimum satisfy idempotency. Furthermore, extended continuous t-(co)norms are further investigated on the algebra of fuzzy values. Fuzzy-valued t-(co)norms are induced by arbitrary t-(co)norms, regardless of their continuity, which take extended continuous t-(co)norms defined on the algebra of fuzzy values as special cases.

Walker and Walker [27] pointed out that the algebra of convex normal fuzzy truth values is a distributive lattice. In [12], the lattice of convex normal fuzzy truth values was further studied and proven to be a complete lattice. However, the least upper bound or greatest lower bound of arbitrary subset of the lattice of convex normal fuzzy truth values is too complex. The algebra of fuzzy values was proven to be a complete distributive lattice in [2,20]. In this paper, we continue to study the lattice of fuzzy values and point out that the algebra of fuzzy values is a complete completely distributive lattice (CCD lattice, for short). Moreover, we provide the list of all join-irreducible elements on the algebra of fuzzy values.

Residuated lattices were investigated by Dilworth [3]. There has been substantial researches regarding some specific classes of residuated lattices, such as [7,11]. In [24], L-fuzzy rough sets were proposed and investigated by Radzikowska and Kerre as a generalization of rough sets proposed by Pawlak [22,23], where L is a residuated lattice. L-fuzzy partial order was investigated from the view of L-fuzzy sets theory [19,31,34] and the view of category theory [30,33], which were shown to be equivalent to each other in [29]. It is a problem how to use knowledge mentioned above to investigate the algebra of fuzzy values and fuzzy-valued fuzzy sets. The key to that problem is constructing residuated lattices on the algebra of fuzzy values. In [16], Kawaguchi and Miyakoshi attempted to show that the algebra of fuzzy values is a residuated lattice without extended supremum. However, this method is not correct. In this paper, we revalidate that the algebra of fuzzy values is a residuated lattice. First, we construct residuated lattices based on extended continuous t-norms. Second, we investigate residuated lattices based on fuzzy-valued t-norms induced by left-continuous t-norms. Third, as an application of fuzzy-valued t-norms induced by left-continuous t-norms, fuzzy-valued fuzzy rough sets and fuzzy-valued fuzzy complete lattices are proposed and investigated in fuzzy-valued fuzzy sets from the view of L-fuzzy sets, where L is a residuated lattice on the algebra of fuzzy values.

The contents of the paper are organized as follows. In Section 2, we recall some fundamental concepts and related properties. Especially, we propose type-2 t-norms and t-conorms. In Section 3, we investigate the lattice structures of the algebra of fuzzy values and point out that the algebra of fuzzy values is a CCD lattice. Section 4 constructs fuzzy-valued t-(co)norms induced by arbitrary t-(co)norms regardless of their continuity, respectively. Moreover, we investigate fuzzy-valued fuzzy implications and construct residuated lattices based on fuzzy-valued t-norms induced by left-continuous t-norms. In Section 5, we propose fuzzy-valued fuzzy rough sets and fuzzy-valued fuzzy complete lattices from the view of L-fuzzy sets. Furthermore, we point out that fuzzy-valued fuzzy rough sets are CCD rough sets defined in [1], when the fuzzy-valued fuzzy relations are $\overline{\top}$-fuzzy-valued fuzzy equivalence relations.

2. Preliminaries

Let X and Y be nonempty universes, $\mathscr{M}(X, Y)$ be the set of all mappings from X to Y and I denote the unit interval [0, 1]. $\mathscr{M}(X, Y)$ is denoted as $\mathscr{M}(X)$, if $Y = I$. A function $N : I \to I$ is called an involutive negator, if it is decreasing and satisfies $N(N(x)) = x$ for all $x \in I$. Assume that $*$ and $\star$ are two binary operations on I, then they are said to be *dual* with respect to (w.r.t., for short) N, if for all $x, y \in I$, $N(x * y) = N(x) \star N(y)$.

A fuzzy set A is a mapping from X to I, i.e., $A \in \mathscr{M}(X)$. A is called a *fuzzy truth value*, if $A \in \mathscr{M}(I)$. The two sets $\emptyset$ and X are special elements in $\mathscr{M}(X)$, with $\emptyset(x) = 0$ and $X(x) = 1$ for all $x \in X$, respectively. The order relation on fuzzy sets is defined as $A \subseteq B \Leftrightarrow A(x) \leqslant B(x)$ for all $x \in X$. For all $a, b \in I$, a special fuzzy truth value b/a is defined as

$$(b/a)(x) = \begin{cases} b, & x = a, \\ 0, & x \neq a. \end{cases}$$

If $b = 1$, then b/a is also denoted as $\bar{a}$ for short. An α-cut set of fuzzy set A on X is $A_\alpha = \{x \in X | A(x) \geqslant \alpha\}$ for all $\alpha \in I$. If $\alpha = 1$, then A_α is called the kernel of A and denoted as Ker(A). A strong α-cut set of fuzzy set A is $A_{\overline{\alpha}} = \{x \in X | A(x) > \alpha\}$ for all $\alpha \in I$. If $\alpha = 0$, then $A_{\overline{\alpha}}$ is called the support of A and denoted as Supp(A). The least upper bound of A is denoted by $A^{\sup}$, i.e., $A^{\sup} = \sup_{x \in X} A(x)$.