Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 268 (2015) 127–140

FUZZY sets and systems

www.elsevier.com/locate/fss

Perturbation of bivariate copulas

Radko Mesiar [a], Magda Komorníková [a,*], Jozef Komorník [b]

[a] *Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia*
[b] *Faculty of Management, Comenius University, Odbojárov 10, P.O. BOX 95, 820 05 Bratislava, Slovakia*

Received 3 December 2013; received in revised form 7 March 2014; accepted 21 April 2014

Available online 30 April 2014

Abstract

New types of constructions of bivariate copulas are introduced, discussed and exemplified. Based on a given copula C, we look for its perturbation into another copula C_H, possibly close to C. A special stress is put on the perturbation of the basic copulas M, W, Π. Our results generalize several methods known from the literature, such as the Farlie–Gumbel–Morgenstern copula family, for example. An illustrative example when fitting copulas to real data is also added.

Keywords: Farlie–Gumbel–Morgenstern copulas; Perturbation of product copula; Radially symmetric copula; Construction method for copula

1. Introduction

Copulas, especially their bivariate form, have obtained a growing interest in the last two decades, especially because of their numerous applications. Recall only that a function $C : [0, 1]^2 \to [0, 1]$ is called a (bivariate) copula [24] whenever it is

i) 2-increasing, i.e.,

$$V_C\big([u_1, u_2] \times [v_1, v_2]\big) = C(u_1, v_1) + C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) \geq 0$$

for all $0 \leq u_1 \leq u_2 \leq 1,\ 0 \leq v_1 \leq v_2 \leq 1$ (recall that $V_C([u_1, u_2] \times [v_1, v_2])$ is the C-volume of the rectangle $[u_1, u_2] \times [v_1, v_2]$);

ii) grounded, i.e., $C(u, 0) = C(0, v) = 0$ for all $u, v \in [0, 1]$;

iii) it has a neutral element $e = 1$, i.e., $C(u, 1) = u$ and $C(1, v) = v$ for all $u, v \in [0, 1]$.

For more details, examples and applications we recommend monographs Joe (1997) [13] and Nelsen (2006) [22]. Fitting of an appropriate copula to real data is one of major tasks in application of copulas. For this purpose, a large

* Corresponding author.
E-mail addresses: Radko.Mesiar@stuba.sk (R. Mesiar), Magdalena.Komornikova@stuba.sk (M. Komorníková), Jozef.Komornik@fm.uniba.sk (J. Komorník).

http://dx.doi.org/10.1016/j.fss.2014.04.016

buffer of potential copulas is necessary, preferably parametric families of copulas. Once we know approximately a copula C appropriate to model the observed data, we look for a minor perturbation of C which fit better than C itself. This is, e.g., the case of Farlie–Gumbel–Morgenstern (FGM) class of copulas, all of them being a perturbation of the independence copula Π, $\Pi(u,v)=uv$. Recall that FGM family $(C_\alpha^{FGM})_{\alpha\in[-1,1]}$ of copulas is given by

$$C_\alpha^{FGM}(u,v)=uv+\alpha u(1-u)v(1-v) \tag{1}$$

see [8,11,21].

Several generalizations of FGM approach to perturb the product copula Π can be found in literature, see, for example, [2,3,9,12,18,23].

As another example, consider the comonotonicity copula M, $M(u,v)=\min(u,v)$, and a parametric noise family $H_\alpha:[0,1]^2\to[0,1]$, $\alpha\in[0,1]$ given by $H_\alpha(u,v)=\alpha(\max(u,v)-1)$. Then, for $\alpha\in\,]0,1[$,

$$M_{H_\alpha}(u,v)=\max\bigl(0,M(u,v)+H_\alpha(u,v)\bigr)=\max\bigl(0,(1-\alpha)M(u,v)+\alpha(u+v-1)\bigr)$$

defines a singular copula with support on 3 segments connecting the point $(\frac{\alpha}{1+\alpha},\frac{\alpha}{1+\alpha})$ with vertices $(0,1)$, $(1,1)$ and $(1,0)$ (if $\alpha=0$, $M_{H_0}=M$; if $\alpha=1$, $M_{H_1}=W$ is the countermonotonicity copula given by $W(u,v)=\max(0,u+v-1)$).

Obviously, also some other copulas perturbations have attracted several researchers, adding some noise to a given copula C, compare [1,14]. We have presented one such approach at AGOP 2013 conference [19], and the aim of this paper is a deeper discussion of a general approach covering all mentioned perturbation methods.

For a given copula $C:[0,1]^2\to[0,1]$, we will look for constraints on the noise $H:[0,1]^2\to\Re$ so that the function $C_H:[0,1]^2\to[0,1]$ given by

$$C_H(u,v)=\max\bigl(0,C(u,v)+H(u,v)\bigr) \tag{2}$$

is also a copula. Obviously FGM copulas given by (1) are linked to $C=\Pi$ and $H_\alpha(u,v)=\alpha u(1-u)v(1-v)$ (observe that in this case, no truncation is necessary).

Recall that for each copula C it holds $W\le C\le M$, and thus the formula (2) can be considered also in the form

$$C_H(u,v)=\max\bigl(W(u,v),\min\bigl(M(u,v),C(u,v)+H(u,v)\bigr)\bigr).$$

Our aim was to generalize the approach exemplified in the two above examples, and thus we work in this paper with formula (2).

The paper is organized as follows. The next section is devoted to several examples and discusses the relationship between C and H. In Section 3, we discuss perturbations of the product copula Π. Section 4 deals with some perturbation of the copula M (and related perturbations of the copula W). In Section 5, we discuss perturbations of radially symmetric copulas. Section 6 brings an illustrative example of fitting copulas to real data. Finally, some concluding remarks are added.

2. Perturbation of bivariate copulas

For a fixed copula $C:[0,1]^2\to[0,1]$, consider the function C_H given by (2). To satisfy the groundedness condition of copulas by C_H, necessarily $H(u,0)\le 0$ and $H(0,v)\le 0$ for all $u,v\in[0,1]$. Similarly, $e=1$ is a neutral element of C_H only if $H(u,1)=H(1,v)=0$ for all $u,v\in[0,1]$. The main problem to ensure that C_H is a copula is to guarantee the 2-increasingness of C_H, which depends both on C and H.

Example 1. Consider the Hamacher product copula C^H (it belongs also to Clayton copulas, Ali–Mikhail–Haq copulas, etc.) given by

$$C^H(u,v)=\frac{uv}{u+v-uv}$$

whenever $(u,v)\ne(0,0)$. For $\alpha\in\Re$, let $H_\alpha:[0,1]^2\to[0,1]$ be given by

$$H_\alpha(u,v)=\alpha\bigl(u^2-u^3\bigr)\bigl(v^2-v^3\bigr).$$