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Compatibility of fuzzy power relations ☆

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Abstract

In this paper we study the conditions under which the compatibility of binary fuzzy relations is preserved by the operation of powering. We pay particular attention to fuzzy power algebras based on the cartesian product of fuzzy sets. We show that the preservation of compatibility essentially depends on the underlying structure of truth values $\mathcal{L}$. Among other things we prove that for fuzzy preorders both Hoare-goodness and Smyth-goodness imply compatibility, but the converse is generally true only under the assumption that $\mathcal{L}$ is a complete Heyting algebra.

1. Introduction

The *power structure* of a structure $\mathcal{A}$ (with a universe A) is an appropriate structure defined on the power set $\mathcal{P}(A)$. There are three basic types of power constructions in universal algebra. The first type is a natural generalization of the multiplication of cosets in group theory, where we "lift" an operation f on a set A to the operation f^+ on the power set $\mathcal{P}(A)$. A more general construction, i.e. the powering of relational structures, was introduced by Jónsson and Tarski in [12]. Here, we associate an n-ary operation on $\mathcal{P}(A)$ to any $n+1$-ary relation on A. This construction has proved to be very useful in various areas of algebraic logic and the theory of non-classical logics. In the third construction relations between elements of a set are extended to relations between subsets of that set. There are several different ways to lift a relation from a base set to its power set. A general definition of n-ary power relation was given in [16]. This type of construction is widely used in theoretical computer science, in the context of power domains. A general view on power structures and common ideas underlying different approaches to this topic can be found in [5,2,6].

The concept of the *fuzzy power algebra* is defined in [8] and [4]. Since there are two natural ways to define a *product* of fuzzy sets – the cartesian and the tensor product – we have (at least) two different ways of "lifting" an n-ary operation on a set A to the set $\mathcal{F}(A)$ of all fuzzy subsets of A. If we take the cartesian product as product of fuzzy sets, we obtain fuzzy power algebras of the type $\mathcal{F}^+(\mathcal{A})$; taking the tensor product for product of fuzzy sets leads to fuzzy power algebras of the type $\mathcal{F}^*(\mathcal{A})$. Algebras of the first type are studied in [8]. In that paper, the real unit interval [0, 1], endowed with a continuous t-norm $*$, serves as the structure of truth values. Among other things, the notions of Smyth good and Hoare good fuzzy relation are introduced and investigated. Fuzzy power algebras obtained

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using the tensor product, are studied in [13]. The authors take a complete residuated lattice for the structure of truth values. As emphasized there, “one advantage of the fuzzy power algebra $\mathcal{F}^*(\mathcal{A})$ over $\mathcal{F}^+(\mathcal{A})$ lies in its relationship to extensions of fuzzy relations on A to the power set $\mathcal{F}(A)$”.

The main aim of the present paper is to answer some questions posed in [8], which could give us a clearer picture of the similarities and differences between the two types of fuzzy power algebras. Namely, it is proved in [13] that for any $*$-congruence R of an algebra $\mathcal{A}$, the power relation R^+ is a $*$-congruence of a power algebra $\mathcal{F}^*(\mathcal{A})$. We will prove that an analogous statement holds for power algebras $\mathcal{F}^+(\mathcal{A})$ precisely when the structure of truth values is a complete Heyting algebra. In the second part of the paper we will consider fuzzy preorders. It is proved in [13] that an $\mathcal{L}$-preorder R is $*$-compatible on an algebra $\mathcal{A}$ if and only if R is $*$-Hoare good on $\mathcal{A}$ if and only if R is $*$-Smyth good on $\mathcal{A}$. Analogous problems concerning $\wedge$-compatibility, $\wedge$-Hoare-goodness and $\wedge$-Smyth-goodness, mostly remained open. In the present paper we prove that both $\wedge$-Hoare-goodness and $\wedge$-Smyth-goodness imply $\wedge$-compatibility, but the converse implications will generally hold only under the assumption that $\mathcal{L}$ is a complete Heyting algebra.

The structure of the paper is the following: in Section 2 we will briefly recall some basic properties of complete residuated lattices and basic notions concerning $\mathcal{L}$-sets and $\mathcal{L}$-relations. In Section 3 we give conditions under which the compatibility of a relation is preserved by lifting this relation from an algebra $\mathcal{A}$ to the fuzzy power algebra $\mathcal{F}^+(\mathcal{A})$. In Section 4 we study good quotient relations. For fuzzy preorders we discuss relationships between $\wedge$-compatibility, $\wedge$-Hoare-goodness and $\wedge$-Smyth-goodness.

2. Preliminaries

A *residuated lattice* is an algebra $\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow, 0, 1\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice with the least element 0 and the greatest element 1, $\langle L, *, 1\rangle$ is a commutative monoid, and $*$ and $\rightarrow$ satisfy the adjointness property, i.e. for all $x, y, z \in L$, $x \leq y \rightarrow z$ if and only if $x * y \leq z$. Residuated lattices have been introduced by Ward and Dilworth [15] and they are also known under several other names. A more general definition of a residuated lattice has been used by several authors, see for instance [7]. From their point of view, residuated lattices defined in this paper are *commutative bounded integral residuated lattices*.

If the lattice $\langle L, \wedge, \vee, 0, 1\rangle$ is complete, then $\mathcal{L}$ is a complete residuated lattice. Complete residuated lattices as a structure of truth values were introduced into the context of fuzzy sets and fuzzy logic by Goguen [9]. A thorough information about the role of residuated lattices in fuzzy logic can be found in [10,11,14]. Here are some of the properties of residuated lattices: for all $x, y, z, a, b \in L$

(L1) $x \leq y$ if and only if $x \rightarrow y = 1$;
(L2) $x \rightarrow x = x \rightarrow 1 = 0 \rightarrow x = 1$ and $1 \rightarrow x = x$;
(L3) $x * 0 = 0$;
(L4) $x * y \leq x \wedge y \leq x$;
(L5) $x \leq y \rightarrow x$;
(L6) $x * (x \rightarrow y) \leq y$;
(L7) $(x \rightarrow z) \wedge (y \rightarrow z) = (x \vee y) \rightarrow z$;
(L8) if $a \leq b$ then $a * x \leq b * x$;
(L9) if $a \leq b$ then $x \rightarrow a \leq x \rightarrow b$;
(L10) if $a \leq b$ then $b \rightarrow x \leq a \rightarrow x$;
(L11) $x * (y \vee z) = (x * y) \vee (x * z)$.

If the residuated lattice is complete, then for all $x \in L$ and any $\{y_i : i \in I\} \subseteq L$

(L12) $x * \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x * y_i)$.

The binary operation $\leftrightarrow$ is defined by: $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$. A residuated lattice $\mathcal{L}$ is called a *Heyting algebra* if for all $x, y \in L$, $x \wedge y = x * y$. Note that a residuated lattice $\mathcal{L}$ is a Heyting algebra if and only if $\mathcal{L}$ is *idempotent* i.e. for all $x \in L$, $x * x = x$. A residuated lattice L is called a *BL-algebra* if for all $x, y \in L$, $x \wedge y = x * (x \rightarrow y)$ (divisibility) and $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity). The most applied complete residuated lattices are those with $L = [0, 1]$ (the real unit interval) where $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$ and $*$ is some left-continuous t-norm. It