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Topological systems as a framework for institutions

Jeffrey T. Denniston [a], Austin Melton [b], Stephen E. Rodabaugh [c,*], Sergey A. Solovyov [d,*,1]

[a] *Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA*
[b] *Departments of Computer Science and Mathematical Sciences, Kent State University, Kent, OH 44242, USA*
[c] *College of Science, Technology, Engineering, Mathematics (STEM), Youngstown State University, Youngstown, OH 44555-3347, USA*
[d] *Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology, Technicka 2896/2, 616 69 Brno, Czech Republic*

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Abstract

Recently, J.T. Denniston, A. Melton, and S.E. Rodabaugh introduced a lattice-valued analogue of the concept of institution of J.A. Goguen and R.M. Burstall, comparing it, moreover, with the (lattice-valued version of the) notion of topological system of S. Vickers. In this paper, we show that a suitable generalization of topological systems provides a convenient framework for certain kinds of (lattice-valued) institutions.

Keywords: Adjoint situation; Affine theory; Comma category; Elementary institution; Localification and spatialization procedure; Topological institution; Topological space; Topological system; Variety of algebras

1. Introduction

There exists a convenient approach to logical systems in computer science, which is based in the concept of *institution* of J.A. Goguen and R.M. Burstall [24]. An institution comprises a category of (abstract) signatures, where every signature has its associated sentences, models, and a relationship of satisfaction; this relationship is invariant (in a certain sense) under change of signature. The slogan, therefore, is “truth is invariant under change of notation”. Examples of institutions include unsorted universal algebra, many-sorted algebra, order-sorted algebra, several variants of first-order logic, partial algebra (see, e.g., [23]). More examples can be found in [13, Subsection 3.2]. A number of authors (including the initiators themselves) have proposed generalizations of institutions in various forms as well

* Corresponding authors.
E-mail addresses: jdennist@kent.edu (J.T. Denniston), amelton@kent.edu (A. Melton), serodabaugh@ysu.edu (S.E. Rodabaugh), solovjovs@fme.vutbr.cz (S.A. Solovyov).
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as advanced their theories [25,26,34,35,40]. Moreover, some authors used a purely category-theoretic approach to institutions (see, e.g., [12]).

There exists the concept of *topological system* of S. Vickers [49], which is based in the ideas of geometric logic [50] and intended to provide a common setting for both topological spaces (point-set topology) and their underlying algebraic structures—locales (point-free topology). In particular, S. Vickers presented system spatialization and localification procedures, which created ways to move back and forth between the categories of topological spaces (resp., locales) and topological systems. Recently, the latter concept has gained in interest in connection with lattice-valued topology. In particular, [9,10] introduced and studied the notion of lattice-valued topological system; [27] discovered a convenient relationship between crisp and lattice-valued topology, based in topological systems; and [43,47] studied a lattice-valued analogue of the above-mentioned system spatialization procedure.

In an attempt to find possible relationships between institutions and topological systems, at the 35th Linz Seminar on Fuzzy Set Theory, J.T. Denniston, A. Melton, and S.E. Rodabaugh [11] presented a lattice-valued analogue of institutions, and showed that (lattice-valued) topological systems provide a particular instance of the latter. Moreover, [42] introduced (crisp) *topological institutions*, based in topological systems, the slogan being that "the central concept is the theory, not the formula". To continue this line of study, several authors considered some other institutional modifications (e.g., probability institutions, quantum institutions, etc.) [5,8], motivated by the ideas of quantum logic (in connection with quantum physics).

We notice that some researchers prefer the reverse of the above slogan. While the theory plays an important role in building logic, the terms should be constructed first (to allow variable substitution), and after that sentences should be constructed on terms to get a sentence functor. The construction of terms through the term monad plays a key role in allowing variable substitution and variable assignment. Having only abstract categories for signatures hides the use of variables and the difference between terms and sentences. Term and sentence construction are two separate processes, which should be revealed together with the process of variable assignment. The readers with the same point of view could look into [18–22] for a particular fuzzy approach to terms and their respective monad. We notice, however, that although the main point of institution theory is exactly to liberate the logic study from explicit variables and substitutions, when needed institution theory has its well established approach to these [13].

The main purpose of this paper is to show that a suitably generalized concept of topological system makes a setting for a particular type of (lattice-valued) institutions, namely, elementary institutions [41,42]. In so doing, we aim at providing a convenient framework for building up the theory of lattice-valued institutions. More precisely, there already exists a well-developed theory of lattice-valued (also many-valued or fuzzy) logic, which has been given a coherent statement by P. Hájek in [28], and which by now has much diversified w.r.t. the algebraic structures (which often constitute a variety) over which the respective fuzzification is done. The concept of institution, however, being a significant part of the crisp logical developments (see, e.g., [13]), has been fuzzified just recently in [11]. With this article, we are going to extend further this fuzzification in a way, which could encompass various lattice-valued frameworks. We achieve this goal with a modification of the affine context of Y. Diers [14–16], which is based in an arbitrary variety of algebras, thereby providing a unifying setting for many possible fuzzifications of institutions (which are to come), each of them based in the favourite variety of it's authors (e.g., a variety of residuated lattices). The main advantage of such a unifying setting is the fact that every statement, which is proved in the affine framework (namely, for all varieties) will be valid for each particular fuzzification (namely, for each particular variety).

2. Affine systems and their related tools

This section reviews the notions of affine systems and spaces, as well as their related spatialization and localification procedures (for details, see [46–48]). Since the localification procedure has not yet appeared in the literature in full detail, we provide a more thorough description. We conclude this section with the approach to topological systems, which is motivated by algebraic theories of F.W. Lawvere [33].

A particular remark is helpful w.r.t. the system terminology of this paper. Following the notion of lattice-valued topological system of [9,10], in [47] is provided a more general concept under the name of variety-based topological system, which eventually gave rise to the notion and theory of categorically-algebraic topology [44]. It was subsequently discovered that the latter concept had already been introduced by Y. Diers [14–16] under the name of affine (or algebraic) set, but in a quite different context with no lattice-valued motivation or system notion. As a consequence our generalized topological spaces (resp., systems) are renamed affine spaces (resp., systems).