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ScienceDirect
Fuzzy Sets and Systems 298 (2016) 158–193
FUZZY sets and systems
www.elsevier.com/locate/fss

A graded approach to cardinal theory of finite fuzzy sets, part I: Graded equipollence ☆

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Received 10 April 2014; received in revised form 7 August 2015; accepted 10 August 2015

Available online 18 August 2015

Abstract

In this article, we propose a fuzzy class relation assigning to each pair of finite fuzzy sets a degree to which they are equipollent, which indicates that they have the same number of elements. The concepts of fuzzy sets and fuzzy classes in the class of all sets (in ZFC) are introduced, and several standard relations and constructions, such as the fuzzy power set and exponentiation, are defined. A functional approach to the cardinal theory of finite fuzzy sets based on graded equipollence is shown, and a relation to generalized cardinals and Wygralak's cardinal theory of finite fuzzy sets defined over triangular norms is demonstrated.

Keywords: Residuated-dually residuated lattice; Finite fuzzy sets; Generalized cardinal; Graded equipollence; Fuzzy cardinality

1. Introduction

In the classical set theory, we can recognize two approaches to the cardinality of sets. The first one is a functional approach that uses one-to-one correspondences between sets to compare their sizes. More precisely, we say that two sets a and b are *equipollent* (*equipotent, equivalent, bijective* or have *the same cardinality*) and write $a \sim b$ if there exists a one-to-one mapping of a onto b (see, e.g., [16,17]). The relation "to be equipollent" is an equivalence on the class of all sets and is called *equipollence* (or *equipotence, equinumerosity* etc.). The second approach is based on the concept of cardinal number expressing the power of a set. A cardinal number of a, denoted by $|a|$, is usually defined as the class of all sets equipollent to a or by the initial ordinal number of this class (supposing the axiom of choice).[1]

The two approaches to the cardinality for fuzzy sets mentioned above have also been studied in the fuzzy set theory. Nevertheless, the attention of the fuzzy audience to a cardinal theory for fuzzy sets is not significant in comparison with the attention paid to cardinal theory of sets, although we have to agree with M. Wygralak, who wrote in [33]:

☆ This work was supported by the European Regional Development Fund in the IT4Innovations Centre of Excellence project (CZ.1.05/1.1.00/02.0070) and the NPUII project LQ1602.

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[1] In this case, when cardinal numbers are expressed as classes, one has to admit classes as objects of the considered theory. For example, this definition is not admissible in Zermelo–Fraenkel axiomatic set theory with the axiom of choice, but it does work in type theory.

http://dx.doi.org/10.1016/j.fss.2015.08.010
0165-0114/

If one likes to introduce the notion of cardinality for fuzzy sets, the main difficulty and difference in comparison with sets lies in the graduation of membership of the elements of a universe **M** in a fuzzy set. Consequently, counting and cardinal calculus under fuzziness become a challenging task which is generally more advanced and complicated than in the case of sets. Motivations for undertaking that task follow from both mathematical theory and multiple applications. Cardinality seems to be one of the most fascinating and enigmatic mathematical aspects of fuzzy sets.

Note that the cardinality of fuzzy sets, especially in the case of finite fuzzy sets, has many applications. Generally, the cardinal theory of fuzzy sets can be used in all situations, where one wants to compare sizes of families of elements satisfying a certain property or to count the number of elements in a family that satisfy a certain property, whereas the property is not precisely specified, which means that one cannot surely decide that the property is true or false for considered elements (e.g., to be *young, tall, clever*, or *rich* for certain families of males and females). For instance, measuring sizes of finite fuzzy sets can be used in fuzzy querying in databases, expert systems, evaluation of imprecisely quantified statements, aggregation, decision making in fuzzy environment, metrical analysis of gray images, calculation of histograms of colors and dominant colors (see [2,4–7,10,25,31,35] and references therein).

The equipollence of (finite) fuzzy sets has been investigated primarily by S. Gottwald [11,12] and M. Wygralak [30,31,33,34] (see also [19]). S. Gottwald proposed a graded approach to the equipollence of fuzzy sets defined using the uniqueness of fuzzy mappings in his set theory for fuzzy sets of higher level. The equivalence classes form *fuzzy cardinals*. Note that fuzzy mappings are defined here as special fuzzy relations. Because this approach is purely theoretical, further substantial research in this direction has not been realized yet. An approach to the non-graded equipollence of fuzzy sets was proposed by M. Wygralak [30] (see also [31]).[2] The equipollence of fuzzy sets is defined by the following equality based on α-cuts of fuzzy sets

$$\bigvee\{\alpha \in [0,1] \mid |A_\alpha| \geq i\} = \bigvee\{\alpha \in [0,1] \mid |B_\alpha| \geq i\} \tag{1}$$

that has to be satisfied for any cardinal number i. One may see that this definition becomes rather trivial for finite fuzzy sets, i.e., fuzzy sets with finite supports, and leads to a mapping between universes under which one fuzzy set is an image of the second fuzzy set. In [33] (see also [34] and [9]), M. Wygralak developed the cardinal theory of finite fuzzy sets defined over triangular norms. To ensure the consistency of the cardinal theory, he introduced an appropriate concept of equipollence of finite fuzzy sets that modifies (1) and respects the requirements of the application of triangular norms. Additionally, a graded generalization of equipollence suggesting that fuzzy sets have approximately the same number of elements has been noted by M. Wygralak in [30,31,33,34], but substantial development of cardinal theory based on this type of equipollence has not been realized yet. In [15] (see also [14]), we proposed a new approach to the equipollence of fuzzy sets over a universe of sets (e.g., a universe of all finite sets, all sets, or a Grothendieck universe). Analogously to Gottwald's approach, a graded equipollence is considered, where the degrees of being equipollent are obtained more simply than in Gottwald's approach. More precisely, this approach is based on a graded one-to-one correspondence between fuzzy sets, where the crisp mappings between universes are considered.

The aim of this first article is to develop a cardinal theory of finite fuzzy sets based on the concept of graded equipollence. The definition of graded equipollence for finite fuzzy sets generalizes the definition proposed in [15] (see also [14]) in the sense of the use of the multiplication of a residuated lattice as an alternative operation to the infimum. We shall show that the graded equipollence with respect to $\odot \in \{\wedge, \otimes\}$ is a fuzzy similarity class relation on the class of all finite fuzzy sets, and well-known statements of the cardinal theory of sets (including the Cantor–Bernstein theorem and the Cantor theorem stating the different cardinalities for sets and their power sets) can mostly be proved in a graded design, where if–then formulas are replaced by the inequalities between the degrees in which the antecedent and consequent are satisfied. For example, the classical formula

$$\text{if } a \sim b \text{ and } c \sim d \text{ then } a \times c \sim b \times d,$$

is expressed by the inequality

$$[A \sim^{\odot} B] \otimes [C \sim^{\odot} D] \leq [A \times C \sim^{\odot} B \times D],$$

[2] Specifically, M. Wygralak defined the equipollence of vaguely defined objects, which includes fuzzy sets.