Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems 298 (2016) 222–237

FUZZY sets and systems

www.elsevier.com/locate/fss

Categories of relations for variable-basis fuzziness

Michael Winter [*,1], Ethan Jackson

Department of Computer Science, Brock University, St. Catharines, Canada

Received 20 June 2014; received in revised form 12 June 2015; accepted 6 July 2015

Available online 10 July 2015

Abstract

Arrow categories establish a categorical and algebraic description of $\mathcal{L}$-fuzzy relations, i.e., relations that use membership values from an arbitrary but fixed complete Heyting algebra $\mathcal{L}$. With other words arrow categories describe the fixed-basis case. In this paper we are interested in the variable-basis case, i.e., the case where relations between different objects may use different membership values. We will investigate the structure of the collection of lattices of membership values within a given Dedekind category. This will lead to a complete characterization of the variable-basis case in this context.

1. Introduction

Allegories and Dedekind categories provide a suitable abstract framework to reason about binary relations [2,12]. In addition to the standard model of binary relations, i.e., the category **Rel** of sets and binary relations, these categorical theories are also suitable for $\mathcal{L}$-fuzzy relations [4]. In such a relation every pair of elements is related up to a certain degree indicated by a membership value from the complete Heyting algebra $\mathcal{L}$. Formally, an $\mathcal{L}$-fuzzy relation R (or $\mathcal{L}$-relation for short) between a set A and a set B is a function $R : A \times B \to \mathcal{L}$. However, the theory of those categories is too weak to express certain notions important in the case of $\mathcal{L}$-fuzzy relations. For example, the notion of crispness cannot be expressed in the language of Dedekind categories. A crisp relation is a relation that assigns either 0 (least element of $\mathcal{L}$) or 1 (greatest element of $\mathcal{L}$) to each pair as a membership value. In order to overcome this deficiency the theory of arrow and Goguen categories has been established as an algebraic and categorical framework to reason about these $\mathcal{L}$-fuzzy relations [14].

The theory of arrow and Goguen categories has been studied intensively [3,14–17,19]. This includes investigations into higher-order fuzziness [20,21], i.e., fuzzy relation that are based on fuzzy membership values. A fuzzy membership value is a function $f : \mathcal{L} \to \mathcal{L}$ indicating for every $x \in \mathcal{L}$ up to which degree $f(x)$ the value x is the membership value in question. In addition to the theoretical studies, the theory has been used to model and specify type-1 and type-2 $\mathcal{L}$-fuzzy controllers [7,18,22].

* Corresponding author. Tel.: +1 905 688 5550x3355; fax: +1 905 688 3255.
E-mail addresses: mwinter@brocku.ca (M. Winter), jackson.ethan.c@gmail.com (E. Jackson).

1 The author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada (283267).

http://dx.doi.org/10.1016/j.fss.2015.07.004

Scalar relations and/or ideal relations can be used to identify the underlying lattice $\mathcal{L}$ of membership values even in the case of abstract arrow or Goguen categories [14,17,18]. However, these categories are uniform – a property that implies that all relations of the category are based on the same lattice $\mathcal{L}$. This means that the theory models the fixed-basis case. Indeed, higher-order fuzziness is modeled via multiple arrow categories. It is based on an abstract definition of a type-2 arrow category over a ground arrow category [21].

In this paper we are interested in the variable-basis case, i.e., the case where multiple lattices are used at the same time. Even though the lattice $\mathcal{L}$ is arbitrary and can be exchanged if we consider $\mathcal{L}$-fuzzy sets and relations as introduced in [4], we are effectively considering a different theory for each $\mathcal{L}$, i.e., we have a variety of fixed basis theories. The first theory that combines fuzziness based on multiple lattices in one theory originated in fuzzy topology. In [10] S.E. Rodabaugh introduced a category whose objects are triples $(X, \mathcal{L}, \tau)$ where τ is an $\mathcal{L}$-fuzzy topology on X. A summary of this approach is given in [11]. However, these theories are based on fuzzy topologies and an appropriate notion of a morphism between those spaces. The morphisms themselves are not fuzzy. We are interested in the situation where relations between different objects may use different membership values. Such a theory is interesting for multiple reasons. First of all, it will provide more insight into the relationship between the fixed- and variable-basis case. Second, it can also serve as foundation for an internal version of higher-order fuzziness rather than an approach involving multiple categories. Last but not least, this theory may also provide the theoretical background for fuzzy controllers that use different membership values within different components.

After recalling some basic definitions in Section 2 we will introduce the notion of a membership basis $\mathfrak{L}$, which mainly consists of a collection of complete Heyting algebras, in Section 3. Given such a membership basis we define a Dedekind category $\mathfrak{L}$-Rel of fuzzy relations using membership values from the lattices $\mathcal{L}$ of $\mathfrak{L}$. In Section 4 we will show that the collection of the lattices of scalar relations of any Dedekind category forms a membership basis. In addition, throughout Sections 3 and 4 we will investigate membership bases that originate from a single lattice. In Section 5 we will define the notion of a weak arrow category generalizing the concept of an arrow category to the variable-basis case. We show that $\mathfrak{L}$-Rel is a weak arrow category and that the property of being uniform relates the variable to the fixed basis case.

2. Mathematical preliminaries

In this section we want to recall some basic notions from lattice, category and allegory theory. For further details we refer to [1,2].

We will write $R : A \to B$ to indicate that a morphism R of a category $\mathcal{R}$ has source A and target B. We will use ; to denote composition in a category, which has to be read from left to right, i.e., $R; S$ means R first, and then S. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A, B]$. The identity morphism on A is written as $\mathbb{I}_A$.

A distributive lattice $\mathcal{L} = \langle L, \sqcap, \sqcup \rangle$ is called a complete Heyting algebra (or frame) iff $\mathcal{L}$ is complete and $x \sqcap \bigsqcup M = \bigsqcup_{y \in M} (x \sqcap y)$ holds for all $x \in \mathcal{L}$ and $M \subseteq \mathcal{L}$. We will denote the least and greatest element of a lattice by 0 and 1 if they exist.

We will use the framework of Dedekind categories [8,9] throughout this paper as a basic theory of relations. Categories of this type are called locally complete division allegories in [2].

Definition 1. A Dedekind category $\mathcal{R}$ is a category satisfying the following:

(1) For all objects A and B the collection $\mathcal{R}[A, B]$ is the universe of a complete Heyting algebra. Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.
(2) There is a monotone operation $^{\smile}$ (called converse) mapping a relation $Q : A \to B$ to $Q^{\smile} : B \to A$ such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q; R)^{\smile} = R^{\smile}; Q^{\smile}$ and $(Q^{\smile})^{\smile} = Q$.
(3) For all relations $Q : A \to B$, $R : B \to C$ and $S : A \to C$ the modular law $(Q; R) \sqcap S \sqsubseteq Q; (R \sqcap (Q^{\smile}; S))$ holds.
(4) For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $X : A \to B$ the following holds: $X; R \sqsubseteq S \iff X \sqsubseteq S/R$.

As mentioned in the introduction the collection of binary relations between sets as well as the collection of $\mathcal{L}$-fuzzy relations (or $\mathcal{L}$ relations, for short) between sets form a Dedekind category. In both categories the lattice operation