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Semiquadratic copulas based on horizontal and vertical interpolation

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Abstract

We introduce several families of semiquadratic copulas (i.e. copulas that are quadratic in any point of the unit square in at least one coordinate) of which the diagonal and/or opposite diagonal sections are given functions. These copulas are constructed by quadratic interpolation on segments connecting the diagonal, opposite diagonal and sides of the unit square; all interpolations are therefore performed horizontally or vertically. For each family we provide the necessary and sufficient conditions on the given diagonal and/or opposite diagonal functions and two auxiliary real functions to obtain a copula that has these diagonal and/or opposite diagonal functions as diagonal and/or opposite diagonal sections. Just as the product copula is a central member of all families of semilinear copulas based on horizontal and vertical interpolation, it turns out that the Farlie–Gumbel–Morgenstern family of copulas is included in all families of semiquadratic copulas introduced and characterized here.

Keywords: Copula; Quasi-copula; Quadratic interpolation

1. Introduction

Bivariate copulas (briefly, copulas) [14] are binary operations on the unit interval having 0 as absorbing element and 1 as neutral element and satisfying the condition of 2-increasingness, i.e. a copula is a function $C : [0, 1]^2 \to [0, 1]$ satisfying the following conditions:

1. for all $x \in [0, 1]$, it holds that

$$C(x, 0) = C(0, x) = 0, \qquad C(x, 1) = C(1, x) = x;$$

2. for all $x, x', y, y' \in [0, 1]$ such that $x \le x'$ and $y \le y'$, it holds that

$$V_C([x, x'] \times [y, y']) := C(x, y) + C(x', y') - C(x, y') - C(x', y) \ge 0.$$

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$V_C([x,x'] \times [y,y'])$ is called the C-volume of the rectangle $[x,x'] \times [y,y']$. The copulas M and W, defined by $M(x,y) = \min(x,y)$ and $W(x,y) = \max(x+y-1,0)$, are called the Fréchet–Hoeffding upper and lower bounds: for any copula C it holds that $W \le C \le M$. A third important copula is the product copula Π defined by $\Pi(x,y) = xy$.

The diagonal section of a copula C is the function $\delta_C : [0,1] \to [0,1]$ defined by $\delta_C(x) = C(x,x)$. A diagonal function is a function $\delta : [0,1] \to [0,1]$ that satisfies the following conditions:

(D1) $\delta(0) = 0$, $\delta(1) = 1$;
(D2) for all $x \in [0,1]$, it holds that $\delta(x) \le x$;
(D3) δ is increasing;
(D4) δ is 2-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$, it holds that

$$\left|\delta\left(x'\right) - \delta(x)\right| \le 2\left|x' - x\right|.$$

Note that (D4) implies that δ is absolutely continuous, and hence differentiable almost everywhere. The diagonal section δ_C of a copula C is a diagonal function. Conversely, for any diagonal function δ there exists at least one copula C with diagonal section $\delta_C = \delta$. For example, the copula K_δ, defined by $K_\delta(x,y) = \min(x, y, (\delta(x) + \delta(y))/2)$, is the greatest symmetric copula with diagonal section δ [15] (see also [5,7]). Moreover, the Bertino copula B_δ defined by

$$B_\delta(x,y) = \min(x,y) - \min\left\{t - \delta(t) \,\middle|\, t \in \left[\min(x,y), \max(x,y)\right]\right\},$$

is the smallest copula with diagonal section δ. Note that B_δ is symmetric.

Similarly, the opposite diagonal section of a copula C is the function $\omega_C : [0,1] \to [0,1]$ defined by $\omega_C(x) = C(x, 1-x)$. An opposite diagonal function [2] is a function $\omega : [0,1] \to [0,1]$ that satisfies the following conditions:

(OD1) $\omega(0) = 0$, $\omega(1) = 0$;
(OD2) ω is 1-Lipschitz continuous, i.e. for all $x, x' \in [0,1]$, it holds that

$$\left|\omega\left(x'\right) - \omega(x)\right| \le \left|x' - x\right|.$$

Note that (OD2) implies that ω is absolutely continuous, and hence differentiable almost everywhere. The opposite diagonal section ω_C of a copula C is an opposite diagonal function. Conversely, for any opposite diagonal function ω, there exists at least one copula C with opposite diagonal section $\omega_C = \omega$. For instance, the copula F_ω defined by

$$F_\omega(x,y) = \max(x+y-1,0) + \min\left\{\omega(t) \,\middle|\, t \in \left[\min(x,1-y), \max(x,1-y)\right]\right\},$$

is the greatest copula with opposite diagonal section ω [2,13].

Diagonal functions and opposite diagonal functions have been used recently to construct several families of copulas [1,3,4,6,10,16].

Copulas with a given diagonal section are important tools for modelling upper (λ_U) and lower (λ_L) tail dependence [1,3] while copulas with a given opposite diagonal section are important tools for modelling upper–lower (λ_{UL}) and lower–upper (λ_{LU}) tail dependence [2]. The above tail dependences are used in the literature to model the dependence between extreme events [18].

In the present paper, we first recall lower semiquadratic copulas [11] and introduce in a similar manner three families of semiquadratic copulas with a given diagonal section. Analogously, we introduce four families of semiquadratic copulas with a given opposite diagonal section. There is a great similarity between the case of a given opposite diagonal section and that of a given diagonal section (see also [2]), which can be explained by the existence of a transformation that maps copulas onto copulas in such a way that the diagonal section is mapped onto the opposite diagonal section and vice versa. In the second part of this paper, we consider the construction of semiquadratic copulas with given diagonal and opposite diagonal sections. Also here, we introduce sixteen families of semiquadratic copulas and, based on a set of transformations (see [8,12]), we classify them into six classes.

This paper is organized as follows. In the next section we introduce lower, upper, horizontal and vertical semiquadratic functions with a given diagonal section and characterize the corresponding families of copulas. In Section 3, we introduce in a similar way lower–upper, upper–lower, horizontal and vertical semiquadratic functions with a given