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Copulas, diagonals, and tail dependence

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Abstract

We present some known and novel aspects about bivariate copulas with prescribed diagonal section by highlighting their use in the description of the tail dependence. Moreover, we present the tail concentration function (which depends on the diagonal section of a copula) as a tool to give a description of tail dependence at finite scale. The tail concentration function is hence used to introduce a graphical tool that can help to distinguish different families of copulas in the copula test space. Moreover, it serves as a basis to determine the grouping structure of different financial time series by taking into account their pairwise tail behavior.

Keywords: Risk aggregation; Copulas; Diagonal section; Risk management; Tail dependence

1. Introduction

Since their introduction in the seminal work by A. Sklar [100,101], copulas have been largely employed in the construction and estimation of multivariate stochastic models. As can be testified by a number of recent investigations and monographs devoted to the topic, copulas have enjoyed a great popularity in different applied sciences, especially when the major issue is to understand/quantify a risk coming from different sources. See, for instance, [9,57,58,67, 72,76,94] and the (many) references therein.

Copulas are the functions that allow to aggregate individual risk factors (usually, expressed in terms of random variables) into one global risk output. Generally, such global risk is the multivariate probability distribution function coupling the individual (one-dimensional) risks by means of a copula, as outlined by Sklar's recipe [100, Theorem 3]. In fact, it is known that, given a continuous multivariate random vector $\mathbf{X} = (X_1, \ldots, X_d)$ such that each X_i is distributed according to a probability law F_i, the joint probability distribution function F of $\mathbf{X}$ may be expressed as

$$F(x_1, \ldots, x_d) = C\big(F_1(x_1), \ldots, F_d(x_d)\big). \tag{1}$$

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See also [14,22] for Sklar's theorem in a non-continuous setting.

In practical applications, however, the global risk is an object that is derivable from the multivariate probability distribution function, but may not directly coincide with it. In quantitative risk management, for instance, the global output can be a number, usually the Value-at-Risk (or any other suitable monetary risk measure) of the weighted portfolio [37,89], or a vector, as in the case of vector-valued risk measures [12]. In environmental sciences, instead, the global risk is usually represented by a number, interpreted as the average return period before the occurrence of a dangerous event [88], or by a curve (the critical layer) in a multidimensional Euclidean space, as in the case of Kendall's return period [47,93,95].

In all these situations, it is remarkable that the global risk depends on the behavior of the multivariate distribution function in some specific regions of its domain, usually corresponding to large quantile exceedances. Translated into the copula space, this implies that the behavior of the copula in specific regions, usually neighborhoods of the vertices of the domain $[0,1]^d$, contains most of the information that are useful to estimate the risk. In Statistics, the latter regions are usually called *tails* of the distributions. Moreover, the specification of the dependence structure in the tails represents one of the main advantages of copula models. As stressed, for instance, by the Basel Committee on Banking Supervision[1]: "The copula approach allows the practitioner to precisely specify the dependencies in the areas of the loss distributions that are crucial in determining the level of risk".

The determination of copulas with a specific tail behavior may hence allow to estimate correctly the region of the distribution that is most needed. As such, several investigations have been carried out during the years from different perspectives, ranging from extreme-value analysis [48,69] to the concept of threshold copulas [55,56].

In this work, we would like to review a specific (yet wide) method to construct (bivariate) copulas with given tail behavior that is grounded on the knowledge of its diagonal section. In copula theory, methods of this type originated in [44,83], even if similar studies have been conducted in the general theory of distribution function in [90–92]. The relevance of such constructions is mainly due to the fact that they are directly connected to a popular measure of tail behavior, which is the *tail dependence coefficient*.

Interestingly, the idea of considering constructions of copulas (or, generally, aggregation functions) with given diagonal section originated in preliminary works about triangular norms [97, Chapter 5]. In particular, one question was whether the continuity of the diagonal section implies the continuity of the associated Archimedean triangular norm. A related problem in this literature has been to determine whether the knowledge of the diagonal section allows to determine uniquely the corresponding Archimedean triangular norms [2] (see also [39,78,79]). Such problems, which have their origin in the theoretical understanding of the behavior of associative functions, have now found practical implications in copula estimation, as testified in [19,51,98].

The main goal of this paper is twofold. First, we review some basic facts about copulas with given diagonal section and discuss possible related problems, showing how the use of recent analytical methods to deal with copulas may help in proving novel results. Secondly, we present some graphical tools about copulas with given diagonal section that can assist the decision maker to choose the relevant copula for the problem at hand, especially when, due to data scarcity, classical goodness-of-it techniques [40] may not be efficient.

The paper is organized as follows. Section 2 presents some known and complementary results about copulas with given diagonal section. Section 3 presents the tail concentration function as a tool to get a better idea about the tail behavior of different copulas. Section 4, instead, introduces some novel tools to detect different tail behavior. One tool proposes a tail-dependent 2D visualization of the copula test space that can be associated to a given dataset. The other one introduces a clustering procedures to grouping time series. Finally, Section 5 concludes.

2. Diagonal sections of copulas: basic properties with complements

Throughout the paper we adopt basic definitions and properties about copula theory, which can be found, for instance, in [35,82].

Let $\mathbb{I} := [0,1]$. For any function $F : \mathbb{I}^d \to \mathbb{R}$ we denote by δ_F its *diagonal section* given by $\delta_F(t) = F(t,t,\dots,t)$ for all $t \in \mathbb{I}$. The natural question is whether, given a suitable function $\delta : \mathbb{I} \to \mathbb{I}$, we may obtain a d-dimensional copula whose diagonal coincides with δ. The full characterization of the problem is provided in [13, Proposition 5.1] and it is reproduced here.

[1] Developments in Modelling Risk Aggregation, October 2010, http://www.bis.org/publ/joint25.htm.