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Additive generators of copulas

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Abstract

In this study, we discuss additive generators of copulas with a fixed dimension $n \geq 2$ and additive generators that yield copulas for any dimension $n \geq 2$. We review the reported methods used to construct additive generators of copulas, and we introduce and exemplify some new construction methods.

Keywords: Additive generator; Archimedean copula; Copula; Joint distribution function

1. Introduction

Since their introduction by Sklar in [24], copulas have become an important tool for modeling the stochastic dependence of random vectors and thus the modeling of real data, which can be viewed as outcomes of some n-dimensional random experiment, $n \geq 2$. Thus, copulas can be considered to be basic tools in statistics, but also in related sciences, including economics, information sciences, and sociology. We recall that copulas aggregate one-dimensional marginal distribution functions into n-dimensional joint distribution functions. As a typical example, we recall the case of independent random variables where the stochastic dependence is captured by the product copula Π and the joint distribution function is simply the product of the corresponding continuous marginal one-dimensional distribution functions.

From an axiomatic viewpoint, a function $C: [0,1]^n \to [0,1]$ is called a (n-dimensional) copula whenever it satisfies the boundary conditions (C1) and it is an n-increasing function (C2), as follows.

(C1) $C(x_1, \dots, x_n) = 0$ whenever $0 \in \{x_1, \dots, x_n\}$, i.e., 0 is an annihilator of C, and $C(x_1, \dots, x_n) = x_i$ whenever $x_j = 1$ for each $j \neq i$ (i.e., 1 is a neutral element of C),

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(C2) For any $\mathbf{x}, \mathbf{y} \in [0,1]^n$, $\mathbf{x} \le \mathbf{y}$ (i.e., $x_1 \le y_1, \dots, x_n \le y_n$), it holds that

$$V_C([\mathbf{x},\mathbf{y}]) = \sum_{\varepsilon \in \{-1,1\}^n} \left(C(\mathbf{z}_\varepsilon) \prod_{i=1}^{n} \varepsilon_i \right) \ge 0,$$

where $\mathbf{z}_\varepsilon = (z_1^{\varepsilon_1}, \dots, z_n^{\varepsilon_n})$, $z_i^1 = y_i$, $z_i^{-1} = x_i$.

Note that $V_C([\mathbf{x},\mathbf{y}])$ is called the C-volume of the rectangle $[\mathbf{x},\mathbf{y}]$.

As mentioned earlier, the main interest in copulas is due to Sklar's theorem [24]: for a random vector $Z = (X_1, \dots, X_n)$, $F_Z: R^n \to [0,1]$ is a joint distribution of Z if and only if there is a copula $C: [0,1]^n \to [0,1]$ such that

$$F_Z(x_1, \dots, x_n) = C\big(F_{X_1}(x_1), \dots, F_{X_n}(x_n)\big), \tag{1}$$

where $F_{X_i}: R \to [0,1]$ is a distribution function related to the random variable X_i, $i = 1, \dots, n$. The copula C in (1) is unique whenever the random variables are continuous. For more details of copulas, we recommend [11] and [21].

A highly prominent class of binary copulas is the class of Archimedean copulas characterized by the associativity of C and the diagonal inequality $C(x,x) < x$ for all $x \in]0,1[$. Note that although Archimedean copulas are necessarily symmetric, i.e., they can model the stochastic dependence of exchangeable random variables (X,Y) only, they comprise most of the copula families employed in financial, hydrological, and other application areas. For fitting purposes, these copulas are used in most of the software systems that deal with copulas, such as [1,9,28,29]. The popularity of Archimedean copulas is explained by their representation using one-dimensional functions, which are generally called additive generators of (binary) copulas. This crucial result is attributed to Moynihan [20].

Theorem 1. *A function $C: [0,1]^2 \to [0,1]$ is an Archimedean copula if and only if there is a convex strictly decreasing function $f: [0,1] \to [0,\infty]$, $f(1) = 0$, such that*

$$C(x,y) = f^{(-1)}\big(f(x) + f(y)\big), \tag{2}$$

where the pseudo-inverse $f^{(-1)}: [0,\infty] \to [0,1]$ is given by

$$f^{(-1)}(u) = f^{-1}\big(\min(u, f(0))\big).$$

The function f is called an additive generator of the copula C and it is unique up to a positive multiplicative constant.

We denote $\mathcal{F}_2$ as the class of all additive generators of the binary copulas characterized in the theorem above. Many families of these generators can be found in numerous previous studies, such as [11,13,21]. Several studies have been devoted to methods for constructing additive generators, which we review in Section 3. We also recall an important link between additive generators of copulas and positive distance functions based on the Williamson transform, as observed and discussed by McNeil and Nešlehová in [17].

However, copulas of higher dimensions can also be generated using additive generators. Thus, previous studies inspired us to review the known details for additive generators of copulas and to introduce some new methods for generating them. This paper is organized as follows. In the next section, we summarize known results for additive generators of n-ary copulas (copulas of any dimension). In Section 3, we review some previously reported methods for constructing additive generators of copulas and we also propose a new construction method based on the Williamson transform (see [17]). Section 4 proposes some new construction methods and we present examples. Finally, we provide some concluding remarks.

2. Additive generators of copulas

For any binary Archimedean copula $C: [0,1]^2 \to [0,1]$ generated by an additive generator $f: [0,1] \to [0,\infty]$, C is also a triangular norm [13,20,23] and thus it can be extended univocally to an n-ary function (we retain the original notation for this extension) $C: [0,1]^n \to [0,1]$ given by

$$C(x_1, \dots, x_n) = f^{(-1)}\left(\sum_{i=1}^{n} f(x_i)\right). \tag{3}$$