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A probabilistic approach to the arithmetics of fuzzy numbers

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Abstract

An alternative look on fuzzy numbers based on two random variables and their summation and multiplication is introduced. This approach to summation covers the standard Zadeh's extension principle and triangular norm-based approach, however, it is more general. We illustrate it on some examples. Moreover, copula-based summation preserving the class of triangular (trapezoidal) fuzzy numbers is discussed.

Keywords: Copula; Fuzzy arithmetics; Fuzzy number; Triangle function; Triangular norm

1. Introduction

Fuzzy numbers and the related arithmetics were thoroughly studied since 1975 [13]. For an overview of the most important definitions and results we recommend [19,6].

Definition 1. Consider a fuzzy subset A of $\mathbb{R} =]-\infty, \infty[$. Let μ_A be its membership function. A is *a fuzzy number* whenever

- it is normal, i.e., $\mu_A(x_0) = 1$ for some $x_0 \in \mathbb{R}$,
- function μ_A is upper semi-continuous and quasi-concave, i.e.,

$$\bigcup_{\alpha>\beta}\{x \in \mathbb{R} \mid \mu_A \geq \alpha\} = \{x \in \mathbb{R} \mid \mu_A \geq \beta\} \quad \text{for all } \alpha, \beta \in [0, 1], \quad \text{and}$$

$$\mu_A\big(\lambda x + (1-\lambda)y\big) \geq \min\big(\mu_A(x), \mu_A(y)\big) \quad \text{for all } x, y \in \mathbb{R} \text{ and } \lambda \in [0, 1],$$

- it has bounded support, i.e., there are $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ such that

$$\mu_A(x) = 0 \quad \text{whenever } x < x_1 \text{ or } x > x_2.$$

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The majority of applications dealing with fuzzy numbers processing deals with their summation and multiplication by reals (in most cases, multiplication of non-negative fuzzy numbers is considered), see, e.g. [6,19]. Therefore we restrict our considerations just to the above mentioned processing of fuzzy numbers.

In any type of fuzzy arithmetics, multiplication by a real constant $c \in \mathbb{R}$ gives either 0 if $c = 0$, or, if $c \neq 0$ then $\mu_{cA}(x) = \mu_A(\frac{x}{c})$. Similarly, concerning $c + A$, we have $\mu_{c+A}(x) = \mu_A(x - c)$.

However, for the processing of proper fuzzy numbers, several approaches have been proposed, so far. They are based on the Zadeh extension principle from [27], and its modification based on a given (left-continuous) triangular norm (t-norm) $\mathbf{T} : [0,1]^2 \to [0,1]$ (i.e., an associative, commutative and monotone binary operation on [0, 1] with neutral element $e = 1$; for more details see [15]).

Note that there are also alternative definition of fuzzy numbers, using the distribution functions, or survival functions, see [14,18,21], but still exploiting formula (3) below. Observe that also Chanas with co-authors [5,4] has already exploited a representation of fuzzy numbers by means of distribution functions (of independent random variables) when dealing with the problem of simulation of values of fuzzy variables.

Inspired by the above mentioned alternative definitions, we offer in this contribution an alternative look on fuzzy numbers and their processing.

The contribution is organized as follows. In the next section, we recall some approaches to the fuzzy number processing known from the literature. In Section 3 we propose a probabilistic look on fuzzy numbers as an ordered pair of random variables $A = (X, Y)$, and propose a probabilistic approach to summation and multiplication of fuzzy numbers. In Section 4, several examples are introduced. Finally, concluding remarks are added.

2. Triangular norm-based processing of fuzzy numbers

In 1975, Zadeh [27] proposed a method for extending real functions to functions acting on fuzzy numbers. This method is known also as the extension principle, and in the case of summation, it results into the next formula, valid for any two fuzzy numbers

$$\mu_{A+B}(z) = \sup\{\min(\mu_A(x), \mu_B(z-x)) \mid x \in \mathbb{R}\}. \tag{1}$$

Observe that considering crisp real $c \in \mathbb{R}$, $\mu_c(x) = \begin{cases} 1 & \text{if } x=c \\ 0 & \text{otherwise} \end{cases}$, (1) results into $\mu_{c+A}(x) = \mu_A(x-c)$ for each fuzzy number A. Note also that $\mu_{A+A}(x) = \mu_A(\frac{x}{2})$ which correspond to $\mu_{cA}(x) = \mu_A(\frac{x}{c})$ valid by the extension principle for each $c \neq 0$. Similarly, the product of fuzzy numbers A and B is given by

$$\mu_{A \cdot B}(z) = \sup\{\min(\mu_A(x), \mu_B(y)) \mid xy = z\}. \tag{2}$$

Note that a related concept for fuzzy quantities, i.e., fuzzy subsets of reals, which can be seen as distribution functions (survival functions) was studied by Klement in [14] (by Lowen in [18] and Rodabaugh in [21]). In their approach, the quantile functions related to the involved distribution functions are considered, and then the summation (multiplication of non-negative) fuzzy numbers simply corresponds to the summation (multiplication) of the corresponding quantile functions. Note that, this approach perfectly fits the formulae (1) and (2).

A generalization of Zadeh's extension principle based on a given triangular norm $\mathbf{T}$ was proposed by Dubois and Prade [7], for an overview we recommend [6]. $\mathbf{T}$-based addition $\boxplus_{\mathbf{T}}$ of fuzzy numbers is defined as follows.

Definition 2. Let A, B be fuzzy numbers and $\mathbf{T}$ a triangular norm. Then the $\mathbf{T}$-sum $D = A \boxplus_{\mathbf{T}} B$ has the membership function

$$\mu_D(z) = \sup\{\mathbf{T}(\mu_A(x), \mu_B(z-x)) \mid x \in \mathbb{R}\}. \tag{3}$$

Similarly, the product $\boxdot_{\mathbf{T}}$ is defined by

$$\mu_{A \boxdot_{\mathbf{T}} B}(z) = \sup\{\mathbf{T}(\mu_A(x), \mu_B(y)) \mid xy = z\}. \tag{4}$$

Note that these operations are not associative, in general. Moreover, it may also happen that $A \boxdot_{\mathbf{T}} A \neq 2A$. For example, consider $\mu_A(x) = \max(1 - |x|, 0)$, and a left-continuous t-norm $\mathbf{T}$ given by