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Robustness of fuzzy connectives and fuzzy reasoning with respect to general divergence measures

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Abstract

This paper discusses the robustness of fuzzy connectives and fuzzy reasoning with respect to general divergence measures. First of all, the concept of DF-metric is proposed. Secondly, several DF-metrics are introduced as well as their properties and some inequalities about them. Then a formula of divergence measure composed by DF-metric is presented. Finally, based on the proposed divergence measures, the concept of perturbations of fuzzy sets is extended. According to the extended concept, the perturbation parameters raised by various fuzzy connectives are studied and the perturbations of fuzzy reasoning are also investigated.

Keywords: Dissimilarity function; Biresiduation; Divergence measure; DF-metric; Perturbation; Robustness; Fuzzy reasoning

1. Introduction

Since Zadeh [42–44] introduced the so-called Compositional Rule of Fuzzy Inference (briefly, CRI) method, various methods of fuzzy reasoning have been proposed [10,14,20,22,26,32]. Robustness is one of the important evaluation criteria for fuzzy reasoning methods.

In fuzzy literature there exist many approaches to evaluating the robustness of fuzzy reasoning. In [41], Ying estimated maximum and average perturbation parameters for various fuzzy reasoning methods. In [8,9], Cai presented δ-equalities with respect to algebraic operators, fuzzy relations, fuzzy modus ponens (briefly, FMP) and fuzzy modus tollens (briefly, FMT). Then Cai and Zhang [10,45] applied control principles to fuzzy reasoning and proposed quantitative robustness measures. In [11], Cheng and Fu evaluated the upper and lower bounds of the output error affected by perturbation parameters of the input under the CRI method. In [18], Georgescu discussed the robustness of fuzzy reasoning based on analysis of δ-equalities and perturbations of fuzzy sets. Nguyen et al. [35] and Li et al. [27] estimated the robustness of fuzzy reasoning through investigating the robustness of fuzzy connectives and presented some methods for finding the most robust elements in classes of fuzzy connectives. In [28], we used a concept similar to the

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modulus of continuity to characterize the robustness of fuzzy connectives and presented robustness results for various fuzzy connectives and the CRI method.

For a fuzzy reasoning method, if small disturbances of input always cause small changes of output, then we say that the method has a good behavior of robustness. Under this circumstance, one may ask what do the items "small disturbances" and "small changes" mean? To answer this issue, several concepts have been proposed, which can be used to characterize the approximate equalities or perturbations of fuzzy sets. Pappis [36] introduced the concept of proximity measure to estimate the similarity between fuzzy sets. Hong and Hwang [21] proposed the α-similarity of fuzzy variables. Cai [8,9] examined δ-equalities of fuzzy sets based on the Chebyshev distance. Ying [41] proposed concepts of maximum and average perturbation parameters of fuzzy sets based on the Chebyshev distance. Dai et al. [12] investigated perturbations of fuzzy sets and fuzzy reasoning on the basis of the normalized Minkowski distance. Jin et al. [24] discussed perturbations of fuzzy sets based on logical equivalence measures. Bustince et al. [4,5,7] presented some measures to settle equalities between fuzzy sets. Based on regular pairs derived from left-continuous t-norms, Wang and Duan [40] defined a divergence measure to evaluate the robustness of the Triple I Inference Method. By comparing and analyzing the above-mentioned concepts, we find out that the differences between them are largely due to the underlying similarity measures or divergence measures adopted.

In [29], we used fuzzy equivalencies to construct similarity measures. In [6], Bustince et al. constructed divergence measures through restricted dissimilarity functions. In [30], we introduced the concept of dissimilarity function, which is a generation of the restricted dissimilarity function and can be used to construct divergence measures. Let T be a t-norm with E_T being the biresiduation of T. Then we call $d_T = 1 - E_T$ a dissimilarity function obtained from the biresiduation. Furthermore, if d_T is a metric on [0, 1], then we call it a DF-metric. In [40], some divergence measures for appraising the robustness of fuzzy reasoning were constructed by DF-metrics. Several DF-metrics were also introduced. In Section 3, we propose some DF-metrics that differ from those proposed in [40]. Then we study properties of DF-metrics and present several inequalities about them. Since DF-metrics are special dissimilarity functions, we use them to construct divergence measures and obtain the following computational formula

$$D(A, B) = \frac{a \sum_{i=1}^{n} d_T(A(x_i), B(x_i))}{nb + (a - b) \sum_{i=1}^{n} d_T(A(x_i), B(x_i))},$$

where d_T is a DF-metric and $a > 0, b > 0$. If $a = b$, then we obtain divergence measures that were used to appraise the robustness of fuzzy reasoning in [40]. In this way Wang and Duan's work [40] becomes special cases of the results presented in this paper. In Section 4, we extend the concept of perturbations of fuzzy sets on the basis of the above-mentioned divergence measures. According to the extended concept, the perturbation parameters raised by various fuzzy connectives are studied and the perturbations of fuzzy reasoning are also investigated.

2. Preliminaries

In this section we briefly recall, without proof, some preliminary definitions and results.

Throughout this paper, X is the universal set; $F(X)$ is the class of all fuzzy sets of X; $A(x)$ is the membership function of $A \in F(X)$; $P(X)$ is the class of all crisp sets of X; A^c is the complement of A with $A^c(x) = 1 - A(x)$ for all $x \in X$.

Definition 1. (See [2].) A continuous, strictly increasing function $\varphi : [0, 1] \longrightarrow [0, 1]$ with boundary conditions $\varphi(0) = 0$, $\varphi(1) = 1$ is called an automorphism of the unit interval.

Definition 2. (See [25].) An associative, commutative and increasing function $T : [0, 1]^2 \longrightarrow [0, 1]$ is called a t-norm if it has the neutral element equal to 1.

An associative, commutative and increasing function $S : [0, 1]^2 \longrightarrow [0, 1]$ is called a t-conorm if it has the neutral element equal to 0.

Fodor and Roubens [17] defined fuzzy equivalence as a binary function on the unit interval in the following way.