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Extension principles for closure operators on fuzzy sets and cuts ☆

Jiří Močkoř

University of Ostrava, Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic

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Abstract

We investigate closure operators defined on various partly ordered sets of fuzzy objects, including sets of all fuzzy sets with values in a complete residuated lattice and sets of all cuts systems, defined in crisp sets and sets with similarity relations, respectively. We proved several extension theorems, under which a closure operator defined on one universe can be extended to a closure operator defined on another universe. We also investigate relationships between continuity of pairs of maps with respect to different closure operators.

1. Introduction

The notions of a closure system and a closure operator are very useful tools in several areas of classical mathematics. Let us mention classical closure operators in topological spaces, closure operators which enable to extend various mathematical structures to better ones (e.g., metric space to a complete metric space, lattice to a complete lattice, etc.), closure operators in various algebraic structures, e.g., topological groups, etc. All these closure operators have a very similar structure. In fact, if U is a universe for our closure operator (e.g., classical sets, set of all metric spaces, set of all lattices, etc.) with some ordering $\leq$ defined on U (e.g., a set inclusion relation $\subseteq$), then a closure operator could be defined as a map $c : U \to U$, satisfying

1. $x \leq c(x)$,
2. $c(c(x)) = c(x)$,
3. $x \leq y \Rightarrow c(x) \leq c(y)$,

for every $x, y \in U$. This led several authors to investigate the closure operators also in the framework of fuzzy set theory. Recall the papers [3,5,4,6,7], where various results about closure operators defined on universes of classical fuzzy sets are presented.

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E-mail address: Jiri.Mockor@osu.cz.

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In the fuzzy set theory and its applications, some other generalizations of classical fuzzy sets are frequently used. Firstly, instead of the unit interval [0, 1] some other structures are used. Most of them are some versions of complete lattices with some additional properties. One of the most frequently used structure is a *complete residuated lattice*, see, e.g., [13] (in some terminology *unital and commutative quantale*, see [14]), i.e. a structure $Q = (L, \wedge, \vee, \otimes, \rightarrow, 0_Q, 1_Q)$. A well known example is the Łukasiewicz algebra $Ł = ([0, 1], \vee, \wedge, \otimes, \rightarrow_Ł, 0, 1)$, where

$$a \otimes b = 0 \vee (a + b - 1)$$

$$a \rightarrow_Ł b = 1 \wedge (1 - a + b).$$

Further, classical fuzzy sets (or fuzzy sets with values in residuated lattice) are defined in sets. But any set A can be considered as a couple $(A, =)$, where $=$ is a standard equality relation defined in A. It is then natural instead of the strict equality relation $=$, to consider some more "fuzzy" equality relation defined in A, which is called *similarity relation*. Hence, instead of a classical set A and a fuzzy set $s : A \rightarrow [0, 1]$, we can use a set with similarity relation (A, δ) (called a Q-set) and a "fuzzy set" $s : (A, \delta) \rightarrow Q$.

In our previous papers [11,12], we introduced a notion of a fuzzy set in sets with similarity relation (A, δ) (the so-called Q-sets), where values of a similarity relation $\delta : A \times A \rightarrow Q$ are from the residuated lattice Q. Q-sets then represent objects in various categories $\mathbf{K}$, with differently defined morphisms. A notion of a fuzzy set in (A, δ) then depends on a category $\mathbf{K}$, i.e., f is a fuzzy set in an Q-set (A, δ) in a category $\mathbf{K}$ (shortly, $f \subsetapprox_{\mathbf{K}} (A, \delta)$), if $f : (A, \delta) \rightarrow (Q, \leftrightarrow)$ is a morphism in $\mathbf{K}$, where $\leftrightarrow$ is the biresiduation operation in Q (= special similarity relation in Q). This formal extension of classical fuzzy sets enables us to develop the fuzzy set theory in any category of Q-sets, with a lot of properties similar to those of classical fuzzy sets. Although such definition of a fuzzy set in a category $\mathbf{K}$ is new, for some concrete examples of a category $\mathbf{K}$, it represents a well known object. In the paper, we are interested in a special category, the category $\mathrm{Set}(Q)$ with morphisms $(A, \delta) \rightarrow (B, \gamma)$, defined as special maps $A \rightarrow B$. In the category $\mathrm{Set}(Q)$, a fuzzy set in an Q-set (A, δ) is any extensional map $s : A \rightarrow Q$, i.e., a map which is well known and frequently used by various authors.

In papers [9,10], we proved, that fuzzy sets in the category $\mathrm{Set}(Q)$ can be represented by some cut systems. Recall that a nested system of α-cuts in A is a system $(C_\alpha)_\alpha$ of subsets of A, such that $C_\alpha \subseteq C_\beta$ if $\alpha \geq \beta$ and the set $\{\alpha \in [0, 1] : a \in C_\alpha\}$ has the greatest element for any $a \in A$. Then for any nested system of α-cuts $\mathbf{C} = (C_\alpha)_\alpha$, a fuzzy set $\mu_{\mathbf{C}} : A \rightarrow [0, 1]$ can be constructed by $\mu_{\mathbf{C}}(x) = \bigvee_{\{\beta : x \in C_\beta\}} \beta$, and, conversely, for any fuzzy set μ in A, a nested system of α-cuts is defined by $C_\alpha = \{x \in A : \mu(x) \geq \alpha\}$. Between nested systems of α-cuts in A and fuzzy sets in A there are some interesting relationships, and from some point of view an investigation of fuzzy sets can be substituted by an investigation of nested systems of α-cuts (see e.g. [1,2]).

These results can be extended in a natural way to the equivalence between classical fuzzy sets and α-cuts. Namely, we proved that any fuzzy set $f \subsetapprox_{\mathrm{Set}(Q)} (A, \delta)$ can be represented by the so-called f-cut system $\mathbf{C} = (C_\alpha)_{\alpha \in Q}$, where C_α are subsets of A with some special properties. If instead of a similarity relation δ, we will consider standard equality relation $=$, these general theorems represent relations between classical fuzzy sets and α-cuts.

A result of all these generalizations is a fact, that instead of one universe for a closure operator (i.e., a set $Z(A)$ of all classical fuzzy sets on a set A) we can consider the following universes:

(1) 2^A = the set of all subsets of A,
(2) $Z(A)$ = the set of all classical fuzzy sets in A, i.e., maps $A \rightarrow Q$,
(3) $F(A, \delta)$ = the set of all fuzzy sets in Q-set (A, δ) in the category $\mathrm{Set}(Q)$,
(4) $D(A)$ = the set of all classical cuts in a set A,
(5) $C(A, \delta)$ = the set of all f-cuts in an Q-set (A, δ) in the category $\mathrm{Set}(Q)$.

The aim of this paper is to investigate a possibility to extend a closure operator defined on one of the above mentioned universe to a closure operator defined on any other universe. We also investigate continuous properties of maps between universes with respect to closure operators.