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L-valued bornologies on powersets

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Abstract

In M. Abel and A. Šostak (2011) [1], the concept of an L-fuzzy bornology was introduced. Actually, an L-fuzzy bornology on a set X is a certain ideal in the family L^X of L-fuzzy subsets of a set X. Here we propose an alternative approach to fuzzification of the concept of bornology. We define an L-valued bornology on a set X as an L-fuzzy subset $\mathcal{B}$ of the powerset 2^X satisfying L-valued analogues of the axioms of a bornology. Basic properties of L-valued bornological spaces are studied. Our special interest concerns L-valued bornologies induced by fuzzy metrics and relative compactness-type L-valued bornologies in Chang–Goguen L-topological spaces.

Keywords: Bornology; L-fuzzy bornology; L-valued bornology; Bounded mapping; Fuzzy (pseudo-)metric; Fuzzy topology; Relative compactness set

1. Introduction and motivation

In order to apply the conception of boundedness to the case of a general topological space, S.T. Hu introduced the notion of bornology [20]:

Definition 1.1. (See [20].) A bornology on a set X is a family $\mathcal{B} \subseteq 2^X$ such that

(1B) $\forall x \in X \quad \{x\} \in \mathcal{B}$;
(2B) if $U \subseteq V$ and $V \in \mathcal{B}$ then $U \in \mathcal{B}$;
(3B) if $U, V \in \mathcal{B}$ then $U \cup V \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called a bornological space and the sets belonging to $\mathcal{B}$ are viewed as bounded in this space.

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Thus a bornology on a set X is an ideal in the powerset 2^X containing all finite sets.

Definition 1.2. (See [20].) Given bornological spaces $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$, a mapping $f : (X, \mathcal{B}_X) \to (Y, \mathcal{B}_Y)$ is called bounded if the image $f(A)$ of every set $A \in \mathcal{B}_X$ belongs to $\mathcal{B}_Y$.

Example 1.3. Important examples of bornological spaces $(X, \mathcal{B})$ are:

- a metric space and its bounded subsets (that is sets with finite diameter);
- a topological space and its relatively compact subsets;
- a uniform space and its totally bounded subsets.

At present the theory of bornological spaces is developed in various directions by many authors. Most of the research involving bornologies, especially at its first stage is done in the context of topological linear spaces, see e.g. the fundamental monographs by H.H. Schaefer [31] and H. Hogle-Nled, [18], and in topological algebras, that is in case when the underlying set besides topology, was endowed with a certain algebraic structure. However, a notable research work, especially in the recent time, is being done in the field of bornologies without referring to the algebraic structure of the underlying set. Specifically, much interest is in the research of the so called bornological universes that is triples $(X, T, \mathcal{B})$ where T is a topology and $\mathcal{B}$ a bornology on a set X. Already in 1966 S.T. Hu proved that the bornological universe is metrizable, that is B is exactly the set of all bounded sets if and only if the original space (X, T) is metrizable, $\mathcal{B}$ has a countable base and for each $B_1 \in \mathcal{B}$ there exists $B_2 \in \mathcal{B}$ such that $\mathrm{cl}(B_1) \subseteq \mathrm{int}(B_2)$. Recently a criteria for a bornological universe to be uniformizable was obtained in [38]. General bornological spaces, in particular, bornological universes play a key role in recent research on convergence structures on hyperspaces [2,3,27], and optimization theory [5]. Yet another field where bornologies on topological spaces proved themselves to be useful is research of the topologies on function spaces induced by these bornologies, see e.g. [4,8,9,29].

Aiming to develop an appropriate concept of bornology in the context of fuzzy sets and fuzzy structures in the paper [1] the concept of an L-bornology (where L is a complete lattice) on a set X was introduced. Namely, an L-bornology there was defined as a special ideal in the L-power-set L^X of L-fuzzy subsets of X.

In the present paper we develop an alternative approach to the "fuzzification" of the concept of bornology. Namely here we define an L-*valued* bornology on a set X as a mapping $\mathcal{B} : 2^X \to L$ satisfying certain L-valued analogues of the properties (1B)–(3B) in Definition 1.1. This mapping in a certain sense determines *the degree of boundedness* $\mathcal{B}(A) \in L$ of a set $A \subseteq X$. Main results of this work without proofs were first presented in [37] and in [36]. In the present work our interest is restricted to the study of bornological-type structures on sets, in particular, on sets endowed with topology, not assuming any algebraic structure on the underlying set. Application and extension of the results and constructions exposed here to the case when the underlying set is endowed also with an algebraic structure will be the subject of our next paper.

The structure of the paper is as follows. After describing in Section 2 the context in which we develop the theory, basic definitions are introduced and discussed in Section 3. Properties of the lattice of L-valued bornologies on a fixed set X are studied in Section 4. Further, in Section 5 we develop a construction of an L-valued bornology on a set X from a family of crisp (that is ordinary) bornologies on the same set. This construction plays the crucial role in Section 7 for constructing [0, 1]-valued bornologies from fuzzy metrics. In Section 6 we consider basic properties of the category of L-valued bornological spaces and their bounded mappings.

The "roots" of the concept of bornology lay in the theory of metric spaces. On the other hand, we introduce the concept of an L-valued bornology just by "fuzzifying" the classical definition of a bornology. Therefore we feel it important to develop a construction of an L-valued bornology in a fuzzy (pseudo-)metric space. This task is being fulfilled in Section 7. Different properties of L-valued bornologies induced by fuzzy (pseudo-)metrics are also studied in Section 7. The most lucid realization of the concept of bornology in general topological spaces is realized, in our opinion, by relatively compact subsets of a topological space. In Section 8 we develop an L-valued analogue of compactness in a (Chang–Goguen) L-fuzzy topological space and apply it for construction of an L-valued bornology in L-fuzzy topological spaces. In the last, Section 9, we expose our position on the general problem of fuzzification of the concept of bornology and discuss some perspectives for the future studies.