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An algorithmic study of relative cardinalities for interval-valued fuzzy sets

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Abstract

The main topic of this paper is the notion of relative cardinality for interval-valued fuzzy sets – its definition, properties and computation. First we define relative cardinality for interval-valued fuzzy sets following the concept of uncertainty modelling given by Mendel's Wavy-Slice Representation Theorem. We expand on previous approaches by considering relative cardinality based on different t-norms and scalar cardinalities and we initiate an investigation of its properties and possible applications. Drawing on the Nguyen–Kreinovich and Karnik–Mendel algorithms, we propose efficient algorithms to compute relative cardinality depending on a chosen t-norm. This seems to be the first such broad and consistent analysis to have been made of relative cardinality for interval-valued fuzzy sets. As a promising application we consider using interval-valued relative cardinality to construct the family of parameterised subsethood measures.

Keywords: Relative scalar cardinality; Interval-valued fuzzy sets; Interval type-2 fuzzy sets; Karnik–Mendel algorithms; Epistemic uncertainty; Subsethood measure; Inclusion index

1. Introduction

The need to model the imprecision and incompleteness of information has given rise to fuzzy set theory and its many extensions. Conventional fuzzy set theory is suitable for handling imprecise (gradual) statements, by allowing degrees of truth other than just true or false. However, it appears to be insufficient in the presence of incomplete (partial) information, when the exact degree of truth cannot be specified. We recognise this kind of situation as involving uncertainty, the "real" truth value being concealed.

Undoubtedly, uncertainty is widespread in real life and practical applications, and cannot be ignored. The phenomenon has been studied for many years, and there are two main approaches to understanding the uncertainty of information [1] – epistemic and ontic. In both of them an uncertain concept is described by a set of its possible rep-

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resentations. In the epistemic approach, without additional knowledge it is impossible to pick the right one among them, although the right one exists. In the ontic one, all representations are equally acceptable and there is therefore no need to distinguish between them. For example let us consider a concept of a monthly salary for assistant professor, described by the interval [$4000, $6000]. Such an interval would have an ontic interpretation if it described the minimum and maximum threshold of the salary for the position of assistant professor in some institution. On the other hand, if this interval was supposed to describe an actual salary of some particular assistant professor, it should be interpreted in an epistemic way. What is important, both interpretations require different processing methods in order to capture the nature of uncertainty involved.

The fuzzy set community, motivated by the importance of uncertainty modelling and processing, has considered some generalisations of classical fuzzy set theory in order to capture the uncertainty factor. Of special interest among these are interval-valued fuzzy sets (IVFS, [2]), Atanassov's intuitionistic fuzzy sets (AIFS [3,4]), interval type-2 fuzzy sets (IT2FS, [5,6]) and general type-2 fuzzy sets (T2FS, [2,7,8]). All of these models have been widely applied in a variety of fields, including medical diagnosis [9–11], approximate reasoning [12,13], classification [14], fuzzy control [15,16], and decision making [17]. Since these approaches provide a more adequate representation of expert knowledge, they frequently outperform classical and type-1 fuzzy approaches. Characterisation of uncertainty associated with these concepts is a separate research problem [18]. In the following we adopt the IVFS approach; however, it should be noted that IVFS and AIFS are mathematically equivalent notions [19]. IVFS theory is able to represent both of the above-mentioned approaches to understanding uncertainty. The ontic one is more common in literature and, moreover, it is often adopted implicitly by the authors. The epistemic approach is still less explored but seems to better reflect many real-life problems and thus there is a need for research in this area. The results presented in this paper, although formally valid for both representations, have been developed especially for epistemic uncertainty.

Adding uncertainty to the field of consideration poses new challenges as regards to how to compare and operate on uncertain objects properly, effectively and without losing information about the amount of uncertainty. Much research has been done in this area, proving that these operations and relations are not just straightforward extensions of their crisp or fuzzy counterparts. Although for practical reasons some measures for uncertain objects are single values (see e.g. some similarity measures [20–22]), it seems to be more adequate to express those measures in an uncertain manner. Such an approach is employed, among others, by Mendel's group, who use intervals to capture the uncertainty of an IT2FS. A deep study of uncertainty measures for IT2FS has been conducted, and centroid, cardinality, fuzziness, variance and skewness were all considered [23,24]. It must be emphasised that, because of the more complex structure of an uncertain objects, the cost of computing uncertain measures is also higher, and thus the construction of such measures may be an algorithmic challenge. The issue of the effectiveness of computing such measures has been addressed by, among others, the Karnik–Mendel algorithms [25,26]. However, there are still many important measures that require further research. The need for a general form of subsethood measure for IVFSs was the direct motivation of our study. Such a problem was also recognised by other researchers, see e.g. recent paper by Takáč [27], who constructed subsethood measures for interval-valued fuzzy sets based on the aggregation of interval fuzzy implications. Our paper is devoted to the approach proposed by Kosko [28], in which subsethood is defined in terms of the relative cardinality of two fuzzy sets. This idea is discussed in Section 3.

The primary objective of this paper is to extend relative cardinality of fuzzy sets to the IVFS case, and to construct effective algorithms for its computation. The notion of relative cardinality itself undoubtedly deserves much attention, as it provides a basis for many important concepts, not only the subsethood measure, but also similarity and entropy measures [29], implication operators [30], quantified sentences [31] and others. As a tool, it is widely applied e.g. in approximate reasoning [12], association rules quality assessment [32], rule-based systems [33], fuzzy control [34,35], group decision making [36,37], etc.

As a separate research problem, relative cardinality has never been given sufficient attention in the context of IVFSs. The need to fill this gap led us to construct t-norm-dependent interval-valued extensions of relative cardinality of fuzzy sets. As a tool for the extension we used the Wavy-Slice Representation Theorem [6], which has the desired ability to preserve the amount of uncertainty of IVFSs. To deal with the high complexity of the processed objects, we extended ideas from the Nguyen–Kreinovich [38] and Karnik–Mendel [25,26] algorithms. We showed that the problem can be solved efficiently in the general case (for any t-norm). Never before has such a comprehensive analysis been made of relative cardinality for interval-valued fuzzy sets.

The rest of the paper is organised as follows. Section 2 gives some background information about fuzzy sets and IVFS. The third section covers relative cardinality, both in the fuzzy case as well as in the proposed extension to