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Bijective transformations of fuzzy implications – An algebraic perspective

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Abstract

Bijective transformations play an important role in generating fuzzy implications from fuzzy implications. In [Representations through a Monoid on the set of Fuzzy Implications, Fuzzy Sets and Systems, 247, 51–67], Vemuri and Jayaram proposed a monoid structure on the set of fuzzy implications, which is denoted by $\mathbb{I}$, and using the largest subgroup $\mathbb{S}$ of this monoid discussed some group actions on the set $\mathbb{I}$. In this context, they obtained a bijective transformation which ultimately led to hitherto unknown representations of the Yager's families of fuzzy implications, viz., f-, g-implications. This motivates us to consider whether the bijective transformations proposed by Baczyński & Drewniak and Jayaram & Mesiar, in different but purely analytic contexts, also possess any algebraic connotations. In this work, we show that these two bijective transformations can also be seen as being obtained from some group actions of $\mathbb{S}$ on $\mathbb{I}$. Further, we consider the most general bijective transformation that generates fuzzy implications from fuzzy implications and show that it can also be obtained as a composition of group actions of $\mathbb{S}$ on $\mathbb{I}$. Thus this work tries to position such bijective transformations from an algebraic perspective.

Keywords: Bijective transformation; Fuzzy implications; Group action; Equivalence relation; Conjugacy classes; Special property

1. Introduction

Fuzzy implications are one of the important operators in fuzzy logic, both for their theoretical and applicational values. They are binary operations on the unit interval [0, 1] defined as follows:

Definition 1.1. (See [3], Definition 1.1.1.) A function $I\colon [0,1]^2 \longrightarrow [0,1]$ is called a *fuzzy implication* if it satisfies, for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$, the following conditions:

$$\text{if } x_1 \le x_2, \text{ then } I(x_1, y) \ge I(x_2, y)\,, \text{ i.e., } I(\cdot\,, y) \text{ is decreasing,} \tag{I1}$$

$$\text{if } y_1 \le y_2, \text{ then } I(x, y_1) \le I(x, y_2)\,, \text{ i.e., } I(x,\,\cdot) \text{ is increasing,} \tag{I2}$$

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$$I(0,0) = 1\,,\; I(1,1) = 1\,,\; I(1,0) = 0\,. \tag{I3}$$

The set of fuzzy implications will be denoted by $\mathbb{I}$.

Fuzzy implications have many applications in the areas like, *fuzzy control, approximate reasoning, decision making, fuzzy image processing* and *data mining* etc. Hence there is always a need to generate newer fuzzy implications with various properties. Among the many methods, *bijective transformations* are one of the earliest methods of generating fuzzy implications from fuzzy implications. Let Φ denote the set of all increasing bijections on $[0,1]$.

1.1. Bijective transformations of fuzzy implications

The first such bijective transformation was proposed by Baczyński and Drewniak in [1]. For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, the φ-conjugate $I_\varphi \in \mathbb{I}$ is defined as follows:

$$I_\varphi(x,y) = \varphi^{-1}\left(I(\varphi(x),\varphi(y))\right), \qquad x, y \in [0,1]. \tag{1}$$

They further showed that I_φ preserves many of the desirable properties of the fuzzy implication I.

Following this, in [6], Jayaram and Mesiar while studying a class of fuzzy implications, namely, special fuzzy implications (see Definition 4.1), proposed the following transformation of an $I \in \mathbb{I}$:

$$\varphi(I)(x,y) = \varphi(I(x,y)), \qquad x, y \in [0,1]\,, \tag{2}$$

where once again, $\varphi \in \Phi$. Let us term this as the $\varphi(I)$-conjugate of $I \in \mathbb{I}$.

Recently, Vemuri and Jayaram, in [9], proposed the following transformation and obtained hitherto unknown representations of two families of fuzzy implications, viz., the Yager's f- and g-implications.

For a given $I \in \mathbb{I}$ and $\varphi \in \Phi$, the φ-pseudo conjugate $I^\varphi \in \mathbb{I}$ is defined as follows:

$$I^\varphi(x,y) = \varphi(I(x,\varphi^{-1}(y))), \qquad x, y \in [0,1]\,. \tag{3}$$

1.2. φ-Pseudo conjugates and group actions on $\mathbb{I}$

Interestingly, the last of the above transformations, given by (3), not only gives rise to fuzzy implications but also has an algebraic connotation. In fact, (3) can be said to have had its beginnings in a purely algebraic context. In [9], the authors had proposed a monoid structure on the set of all fuzzy implications $(\mathbb{I}, \circledast)$ – see Definition 2.1 – and determined the largest subgroup $\mathbb{S}$ of this monoid. The transformation (3) arose naturally while studying the action of the group $(\mathbb{S}, \circledast)$ on the set $\mathbb{I}$.

Motivated by the fact that any group action on a set partitions the set, in [9], they have defined a relation $\sim_{\mathcal{V}}$ on $\mathbb{I}$ based on the φ-pseudo conjugates in the following manner. Given $I, J \in \mathbb{I}$,

$$I \sim_{\mathcal{V}} J \iff J = I^\varphi \text{ for some } \varphi \in \Phi\,. \tag{4}$$

Further they have shown that $\sim_{\mathcal{V}}$ is an equivalence relation on $\mathbb{I}$. In fact, as shown in [9], Remark 4.6, the $\sim_{\mathcal{V}}$ equivalence classes form precisely the partition obtained from the particular group action of $\mathbb{S}$ on $\mathbb{I}$.

1.3. Motivation of this work

Let us now define the following relations on $\mathbb{I}$ based on the φ-conjugates and the transformation (2). Given $I, J \in \mathbb{I}$ we define

$$I \sim_{\mathcal{B}} J \iff J = I_\varphi \text{ for some } \varphi \in \Phi\,, \tag{5}$$

$$I \sim_{\mathcal{J}} J \iff J = \varphi(I) \text{ for some } \varphi \in \Phi\,. \tag{6}$$

It can be easily seen that both $\sim_{\mathcal{B}}, \sim_{\mathcal{J}}$ are equivalence relations and hence partition the set $\mathbb{I}$ into equivalence classes. This leads one to investigate the following:

Question 1. Do the bijective transformations (1) and (2) have any algebraic connotations?