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Multi- and multi-polar capacities

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Abstract

Aggregation of inputs from different categories generalizes both unipolar (standard) and bipolar aggregation. This aggregation covers two main cases of multi-polar and non-monotone aggregation. Inputs from different categories can be found in classification methods, multi-criteria decision making, game theory and political and sociological sciences. Two main concepts related to aggregation on the unit interval are capacities and aggregation on grid points – Boolean functions. In this paper multi- and multi-polar capacities that extend (bi- and bipolar) capacities are introduced. The position of these capacities is clarified by means of the connection between Boolean (crisp) functions, capacities, aggregation and concepts from other domains of science which are shown for the unipolar, the bipolar and the multi-polar case. The connection between multi- and multi-polar capacities and the concept of games with n players and r alternatives is described.

Keywords: Multi-polar aggregation; Boolean function; Capacity; Bi-capacity; Bipolar capacity; Bi-cooperative game; Generalized bipolar capacity

1. Introduction

Aggregation operators are used in many domains where a group of input values should be represented by one output value. An aggregation operator [2,10,20] is a function working on an arbitrary number of inputs from the unit interval $[0, 1]$, i.e., $A: \bigcup_{n\in\mathbb{N}}[0, 1]^n \longrightarrow [0, 1]$. The bipolar scale allows to deal with positive, supporting information as well as negative, excluding one. Two possible models used to represent bipolarity (see [19]) are: the bipolar univariate model and the unipolar bivariate model. In the former case, the scale is divided in two zones by a neutral point: a positive feeling is associated with the zone above the neutral point and a negative feeling with the zone below this point. Thus the focus here is on the interval $[-1, 1]$ with the central neutral value 0. In some cases, we might be unable to make a synthesis of possibly contradictory stimuli relative to a single viewpoint, which leads to the bivariate unipolar model. In this case an alternative can receive both a positive and a negative evaluation, reflecting contradictory feelings or stimuli. Here the focus is on the scale $[0, 1] \times [-1, 0]$, or equivalently on the scale $[0, 1]^2$. Examples of bipolar aggregation operators include the symmetric and the asymmetric Choquet integral [11,12,30] and their generalization used in the cumulative prospect theory [32].

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Multi-polar aggregation operators (see [24–27]) were introduced for aggregation of rule outputs in fuzzy rule-based classification systems. These operators represent a special case of aggregation of inputs from different classes (categories), i.e., the case when the aggregation is monotone. Multi-polarity also appears in political sciences (multi-polar world) and we will show in this paper that multi-polarity also appears in sociological sciences (in decision making), and in game theory. Similarly as in the bipolar case also in the multi-polar case we distinguish two possible multi-polar input spaces: the multi-polar space corresponds to the multi-polar univariate model, and the extended multi-polar space corresponds to the unipolar multi-variate model.

If we focus on the unit interval we can identify two concepts related to the aggregation on $[0, 1]$: (monotone) Boolean functions and capacities (pseudo-Boolean functions). A Boolean function $f: \{0, 1\}^n \longrightarrow \{0, 1\}$ can be also seen as a set function on $\mathcal{P}(X)$ for $X = \{1, \dots, n\}$. Here $A \in \mathcal{P}(X)$ corresponds to its characteristic function $\mathbf{1}_A = (x_1, \dots, x_n) \in \{0, 1\}^n$, where $x_i = 1$ if $i \in A$ and $x_i = 0$ otherwise.

Definition 1. Let $X = \{1, \dots, n\}$. A set function $\mu: \mathcal{P}(X) \longrightarrow [0, 1]$ is called a capacity (normalized fuzzy measure) if $\mu(\emptyset) = 0$, $\mu(X) = 1$ and if it is monotone with respect to the set inclusion, i.e., $\mu(A) \le \mu(B)$ whenever $A \subseteq B$.

In optimization problems capacities are also called pseudo-Boolean functions. The measure $\mu(A)$ can be interpreted as a weight related to the subset A of inputs. It is evident that if the range of a capacity μ is $\mathrm{Ran}(\mu) = \{0, 1\}$ then μ is just a monotone Boolean function b with $b(0, \dots, 0) = 0$, $b(1, \dots, 1) = 1$. Similarly, any threshold function applied to a capacity μ yields again a Boolean function. Moreover, we have the following:

Lemma 1. *Let $\lambda_1, \dots, \lambda_n \in [0, 1]$, $\sum_{i=1}^n \lambda_i = 1$ and let $b_1, \dots, b_n: 2^X \longrightarrow \{0, 1\}$ be monotone Boolean functions such that $b_i(0, \dots, 0) = 0$, $b_i(1, \dots, 1) = 1$ for $i = 1, \dots, n$. Then the convex combination $b: 2^X \longrightarrow [0, 1]$ given by*

$$b(A) = \sum_{i=1}^{n} \lambda_i \cdot b_i(A)$$

is a capacity.

Definition 2. A mapping $A: \bigcup_{n \in \mathbb{N}} [0, 1]^n \longrightarrow [0, 1]$ is called an aggregation operator if

(A1) A is non-decreasing, i.e., for any $n \in \mathbb{N}$, $\mathbf{x}, \mathbf{y} \in [0, 1]^n$, $\mathbf{x} \le \mathbf{y}$ it holds $A(\mathbf{x}) \le A(\mathbf{y})$.
(A2) $0, 1$ are idempotent elements of A, i.e., for any $n \in \mathbb{N}$ there is $A(\underbrace{0, \dots, 0}_{n\text{-times}}) = 0$ and $A(\underbrace{1, \dots, 1}_{n\text{-times}}) = 1$.
(A3) For $n = 1$, $A(x) = x$ for all $x \in [0, 1]$.

Assuming any aggregation operator $A: \bigcup_{n \in \mathbb{N}} [0, 1]^n \longrightarrow [0, 1]$, the restriction of A onto $\{0, 1\}^n$ is just a capacity on X. On the other hand, using the Choquet integral [11] the values of a capacity can be interpolated into the aggregation on the whole unit hypercube $[0, 1]^n$.

The same relations as among Boolean functions, capacities and aggregation are valid also for respective concepts in the bipolar and the multi-polar case. This shows us that Boolean functions, capacities and aggregation operators are linked together and therefore it is important to study all three of them. In the bipolar case all the three compounds were already studied (see Section 3).

Our focus is on the multi-polar case. We have started in [28] with a discussion about multi-polar Boolean functions (called multi-polar crisp functions). These functions are related to the univariate multi-polar model, while the extended multi-polar crisp functions related to the multi-variate unipolar model were not yet discussed. We have studied (extended) multi-polar aggregation operators in [24] (multi-polar Choquet integrals based on unipolar capacities), [25] (extended multi-polar t-norms), [27] (multi-polar t-conorms) and [26] (multi-polar averaging operators). In this paper we would like to continue by introduction of (extended) multi-polar capacities and thus extend the respective concepts working on the bipolar scale that were introduced and investigated by Grabisch and Labreuche [17,18] and Greco et al. [21].