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Superadditive and subadditive transformations of integrals and aggregation functions

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Abstract

We propose the concepts of superadditive and of subadditive transformations of aggregation functions acting on non-negative reals, in particular of integrals with respect to monotone measures. We discuss special properties of the proposed transforms and links between some distinguished integrals. Superadditive transformation of the Choquet integral, as well as of the Shilkret integral, is shown to coincide with the corresponding concave integral recently introduced by Lehrer. Similarly the transformation of the Sugeno integral is studied. Moreover, subadditive transformation of distinguished integrals is also discussed.

Keywords: Aggregation function; Superadditive and subadditive transformations; Fuzzy integrals

1. Introduction

The concepts of subadditivity and superadditivity are very important in economics. For example, consider a *production function* $A : \mathbb{R}^n_+ \to \mathbb{R}_+$ assigning to each vector of production factors $\mathbf{x} = (x_1, \ldots, x_n)$ the corresponding output $A(x_1, \ldots, x_n)$. If one has available resources given by the vector $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$, then, the production function A assigns the output $A(\bar{x}_1, \ldots, \bar{x}_n)$. Now suppose that the resources $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n)$ can be divided into $k \in \mathbb{N}$ subgroups of production factors $\bar{\mathbf{x}} = (\bar{x}_1, \ldots, \bar{x}_n) = (x_1^{(1)}, \ldots, x_n^{(1)}) + \ldots + (x_1^{(k)}, \ldots, x_n^{(k)})$. Since the purpose of any production function is to maximize the use of factor inputs in production, one should check if the production output

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$\sum_{i=1}^{k} A(x_1^{(i)}, \dots, x_n^{(i)})$ is greater than $A(\bar{x}_1, \dots, \bar{x}_n)$. More in general, one can be interested in finding the "best" decomposition of the available resources, i.e., we should look to the quantity

$$A^*(\bar{\mathbf{x}}) = \sup\{\sum_{i=1}^{k} A(x_1^{(i)}, \dots, x_n^{(i)}) \mid \sum_{i=1}^{k}(x_1^{(i)}, \dots, x_n^{(i)}) \le \bar{\mathbf{x}}\},$$

provided that $\sum_{i=1}^{k}(x_1^{(i)}, \dots, x_n^{(i)})$ is an allowable (realistic) decomposition of $\bar{\boldsymbol{x}}$. Thus, either $A^*(\bar{\mathbf{x}}) = A(\bar{\mathbf{x}})$ for all $\bar{\mathbf{x}} \in \mathbb{R}_+^n$ or function A^* should be considered – at least ideally – the "real" production function (provided that the range of A^* contains only finite values, i.e., $A^*(\bar{\boldsymbol{x}}) < \infty$ for each $\bar{\boldsymbol{x}} \in \mathbb{R}_+^n$).[1] The condition $A(\bar{\mathbf{x}}) = A^*(\bar{\mathbf{x}})$ for all $\bar{\mathbf{x}} \in \mathbb{R}_+^n$ is equivalent to the superadditivity of the production function A, i.e., for all $\boldsymbol{y}, \boldsymbol{x} \in \mathbb{R}_+^n$ we have $A(\boldsymbol{y}+\boldsymbol{x}) \ge A(\boldsymbol{y}) + A(\boldsymbol{x})$. Analogously, consider the situation of a system of prices represented by the function $A : \mathbb{R}_+^n \to \mathbb{R}_+$ assigning to each bundle of goods $\mathbf{x} = (x_1, \dots, x_n)$ with x_i, $i = 1, \dots, n$, representing the quantity of the i-th item, the corresponding price $A(x_1, \dots, x_n)$. If one wants to buy the bundle $\bar{\mathbf{x}} = (\bar{x}_1, \dots, \bar{x}_n)$, then one can get it at the following price (possibly asymptotically)

$$A_*(\bar{\boldsymbol{x}}) = \inf\{\sum_{i=1}^{k} A(x_1^{(i)}, \dots, x_n^{(i)}) \mid \sum_{i=1}^{k}(x_1^{(i)}, \dots, x_n^{(i)}) \ge \bar{\boldsymbol{x}}\}.$$

Thus either $A(\bar{\mathbf{x}}) = A_*(\bar{\mathbf{x}})$ for all $\bar{\mathbf{x}} \in \mathbb{R}_+^n$ or function A_* becomes the "real" price system considered by economic operators. The condition $A(\bar{\mathbf{x}}) = A_*(\bar{\mathbf{x}})$ for all $\bar{\mathbf{x}} \in \mathbb{R}_+^n$ is equivalent to the subadditivity of the price system A, i.e., for all $\boldsymbol{y}, \boldsymbol{x} \in \mathbb{R}_+^n$ we have $A(\boldsymbol{y} + \boldsymbol{x}) \le A(\boldsymbol{y}) + A(\boldsymbol{x})$.

Observe that in two above examples the superadditivity and the subadditivity of function A were related to its transformations A^* and A_*, respectively. For this reason, it is important to study and discuss these transformations what we shall do in this paper.

For a class $\mathcal{K}$ of some objects, a property $\boldsymbol{p}$ determines a subclass

$$\mathcal{K}_{\boldsymbol{p}} = \{K \in \mathcal{K} \mid K \text{ has property } \boldsymbol{p}\}.$$

Any mapping $\tau : \mathcal{K} \to \mathcal{K}$ is called a transformation (of objects from $\mathcal{K}$), and if $\mathcal{K}_\tau = \{\tau(K) | K \in \mathcal{K}\} = \mathcal{K}_{\boldsymbol{p}}$, and $\tau(K) = K$ for each $K \in \mathcal{K}_{\boldsymbol{p}}$, τ is called a $\boldsymbol{p}$ transformation. Obviously, $\tau \circ \tau = \tau$ for any $\boldsymbol{p}$-transformation τ. Formally, τ can be seen as a projection from $\mathcal{K}$ onto $\mathcal{K}_{\boldsymbol{p}}$.

We recall some typical examples:

- For $\mathcal{K} = \mathcal{M}_{\mathscr{S}}$ the class of monotone measures on a measurable space $(X, \mathscr{S})$, one can consider the superadditivity property. Define a transformation $\tau : \mathcal{M}_{\mathscr{S}} \to \mathcal{M}_{\mathscr{S}}$ by $\tau(m) : \mathscr{S} \to [0, \infty]$,

$$\tau(m)(E) = \sup\{\sum_{i=1}^{k} m(E_i) | (E_i)_{i=1}^{k} \text{ is a measurable partition of } E\}.$$

  Observe that considering the PAN-integral $\int^{PAN}$ introduced in [18], see also [17], $\tau(m)(E) = \int^{PAN} \mathbf{1}_E \, dm$. It is not difficult to check that, taking the property $\boldsymbol{p}$ = superadditivity, then τ is a superadditive transformation.
- For $\mathcal{A}_{[0,1],n}$ the class of n-ary aggregation functions on $[0, 1]$, one can consider the averaging property characterizing idempotent aggregation functions. Then, for the class $\mathcal{K} \subset \mathcal{A}_{[0,1],n}$ of n-ary aggregation functions

[1] To the best of our knowledge, this concept of transformation of the production function from A to A^* is original and not standard in the literature on production functions (see, e.g., [3]). Indeed according to Shephard [13] production function is defined as a relationship between the maximal technically feasible output and the inputs needed to produce that output, that corresponds to what we called "real production function" A^*. However, very often production function is simply defined as a technical relationship between output and inputs without any reference to the assumption that such output has to be maximal with respect to the given inputs (see [12]). In this sense, we can see that, when it is possible to imagine divisibility of the input, the superadditive transformation of the merely technical relationship between output and inputs A gives the "real production function" A^*. Observe that in economics some assumptions are considered on production functions that imply their superadditivity. More precisely, continuity, strict increasing monotonicity, strict quasiconcavity and $A(0) = 0$ are conditions usually assumed on production function. Under these conditions production function is superadditive ([13]; see also Theorem 3.1 in [6]).