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Categorical foundations of variety-based bornology ☆

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Abstract

Following the concept of topological theory of S.E. Rodabaugh, this paper introduces a new approach to (lattice-valued) bornology, which is based in bornological theories, and which is called variety-based bornology. In particular, motivated by the notion of topological system of S. Vickers, we introduce the concept of variety-based bornological system, and show that the category of variety-based bornological spaces is isomorphic to a full reflective subcategory of the category of variety-based bornological systems.

Keywords: Bornological space; Bornological system; Bornological theory; Ideal; Powerset theory; Reflective subcategory; System spatialization procedure; Topological category; Variety of algebras

1. Introduction

The theory of bornological spaces (which takes its origin in the axiomatization of the notion of boundedness of S.-T. Hu [20,21]) has already found numerous applications in different branches of mathematics. For example, the main ideas of modern functional analysis are those of locally convex topology and convex bornology [18]. Additionally (as a link to physics), one could mention the notion of Hausdorff dimension in convex bornological spaces, motivated by the study of the complexity of strange attractors [6].

In 2011, M. Abel and A. Šostak [1] introduced the concept of lattice-valued (but fixed-basis, in the sense of [19]) bornological space, making thereby the first steps towards the theory of lattice-valued bornology. In a series of papers [28,26,27], the present authors made further steps in this direction, taking their inspiration in the well-developed theory of lattice-valued topology. In particular, we presented a variable-basis analogue (in the sense of [31]) of the concept of M. Abel and A. Šostak as well as introduced lattice-valued variable-basis bornological vector spaces (following the pattern of the fuzzy topological vector spaces of [23,24]); found the necessary and sufficient condition for

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the category of lattice-valued bornological spaces to be topological (we notice that all the currently used categories for lattice-valued topology are topological [32]); showed that the category of strict (in the sense of [1]) fixed-basis lattice-valued bornological spaces is a quasitopos (we notice again that the categories for lattice-valued topology fail to have this convenient property); and also provided a bornological analogue of the notion of topological system of S. Vickers [37] (introduced as a common setting for both point-set and point-free topology), starting thus the theory of point-free bornology.

In 2012, S. Solovyov [35] presented a new framework for doing lattice-valued topology, namely, variety-based topology. This new setting was induced by an attempt of S.E. Rodabaugh [32], to provide a common framework for the existing categories for lattice-valued topology through the concept of powerset theory (we notice that there also exists the notion of topological theory of O. Wyler [38,39], which is discussed in [2], and which is compared with the powerset theories of S.E. Rodabaugh in [32]). Unlike S.E. Rodabaugh, however, who tried to find the basic algebraic structure for lattice-valued topology, arriving thus at the new concept of semi-quantale, S. Solovyov decided to allow for an arbitrary algebraic structure, thus arriving at varieties of algebras. Briefly speaking, noticing that most of the currently popular approaches to lattice-valued topology are based in powersets of the form L^X, where X is a set and L is a complete lattice (possibly, with some additional axioms and/or algebraic operations), S. Solovyov decided to base his topology in powersets of the form A^X, in which the lattice L is replaced with an algebra A from an arbitrary variety (in the extended sense, i.e., allowing for a class of not necessarily finitary operations as in, e.g., [7,30]). It appeared that such a general variety-based approach is still capable of producing some convenient results, e.g., all the categories of variety-based topological spaces are topological, and, moreover, one has a good concept of topological system, the category of which contains the category of variety-based topological spaces as a full (regular mono)-coreflective subcategory. Even more, variety-based approach to systems incorporates (apart from the lattice-valued extension of topological systems of J.T. Denniston, A. Melton and S.E. Rodabaugh [10,11]), additionally, state property systems of D. Aerts [3–5] (which serve as the basic mathematical structure in the Geneva–Brussels approach to foundations of physics) as well as Chu spaces of [29] (which eventually are nothing else than many-valued formal contexts of Formal Concept Analysis [12]; we notice, however, that Chu spaces were introduced in a more general form by P.-H. Chu in [8]).

The purpose of this paper is to provide a variety-based setting for lattice-valued bornology. In particular, we introduce both variety-based bornological spaces and systems, and show that the category of the former is isomorphic to a full reflective (unlike coreflective in the topological case) subcategory of the latter. Both spaces and systems rely on powerset theories, which, however, are radically different from those of topology. More precisely, one of the main difficulties for introducing a variety-based approach to bornology lies in the fact that while classical topologies are just subframes [22] of powersets, bornologies are lattice ideals of powersets. Thus, while the switch from subframes to subalgebras is immediate, one needs a good concept of ideal in universal algebras, to switch from lattice ideals to algebra ideals. In this paper, we rely on the concept of algebra ideal of [15] (which eventually goes back to an earlier paper of A. Ursini). We also notice that motivated by, e.g., [36], we distinguish between powerset theories and bornological theories. The main reason for this in the topological setting of [36] is the fact that while powersets are Boolean algebras, topologies are just subframes, i.e., when dealing with topologies, one "forgets" a part of the available algebraic structure.

If properly developed, the theory of variety-based bornology could find an application in cancer-related research. More precisely, at the end of Section 7 in [16], its authors mention that a "systematic study of continuous spectrum of fractal dimension can put more light on several fractal organisms/objects observed in tissues of cancer patients". Moreover, it appears that "in practical applications, we can meet a bornological space instead of a metric one" [16]. Our papers [26–28] together with the present one, could provide a starting point for building a convenient framework for dimension theory for variety-based bornological spaces (e.g., for the already mentioned concept of Hausdorff dimension for bornological spaces of [6]).

This manuscript is based in both category theory and universal algebra, relying more on the former. The necessary categorical background can be found in [2,17]. For algebraic notions, we recommend [9,14,30]. Although we tried to make the paper as much self-contained as possible, it is expected from the reader to be acquainted with basic concepts of category theory, e.g., with that of reflective subcategory.